A Quantitative Survey Of Two Eigenvalue Bounds Of Matrix Polynomials

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Objective

We examine two improvements on the Cauchy and Pellet radii of matrix polynomials. These improvements may be significant but no comprehensive comparison has been carried out to date to verify their effectiveness. Our goal is to do precisely that for matrix polynomials arising from real-world applications.

Introduction

The polynomial eigenvalue problem is to solve $P(z)v = 0$ for a nonzero eigenvector v and corresponding eigenvalue z, where the matrix polynomial P is given by

$$P(z) = A_n z^n + A_{n-1} z^{n-1} + \cdots + A_0, \qquad (1)$$

with $A_j \in \mathbb{C}^{mxm}$ for $j = 0, 1, ..., n$. We will use this definition for $P(z)$ henceforth.

Matrix polynomials arise in many engineering fields such as structural dynamics, fluid mechanics, and vibration analysis. Their eigenvalues often have physical significance. For example, if a matrix polynomial is constructed to model the vibrations of a bridge affected by wind, the eigenvalues correspond to the natural resonance frequencies and can be used to build efficient solutions to minimize resonance in the bridge [4]. While the degree of these matrix polynomials is typically low (2-4), the coefficients can sometimes be quite large (hundreds to thousands of rows and columns).

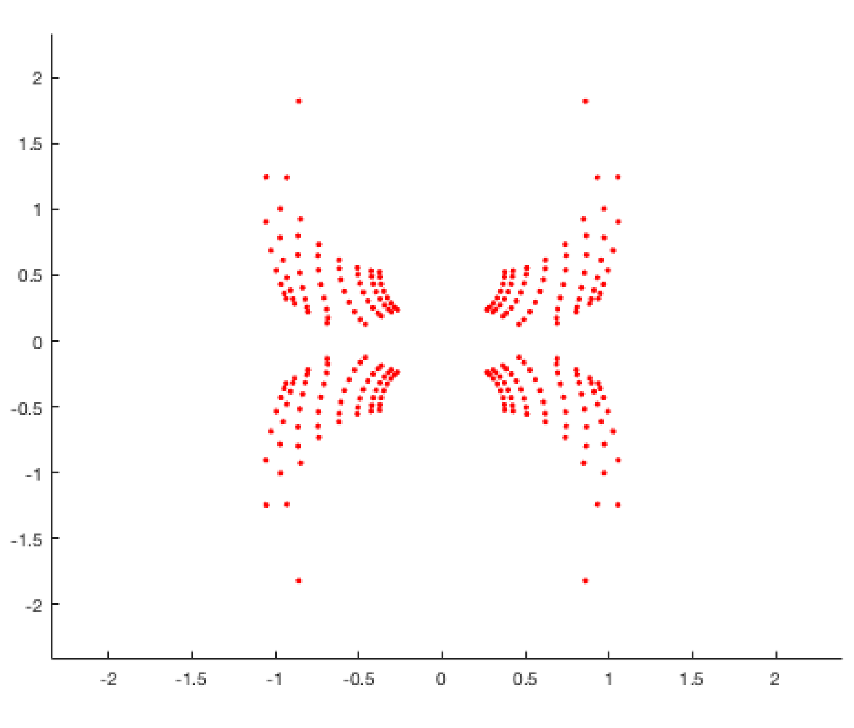

Figure 1: Example of eigenvalues of the 60x60 butterfly problem.

Cauchy Bound Results

Generalized Cauchy Theorem. All eigenvalues of $P(z)$, as defined in (1), lie in $z \leq R$ when A_n is nonsingular, and lie in $z \geq r$ when A_0 is nonsingular, where R and r are the unique positive roots of $||A_n^{-1}||^{-1}x^n - ||A_{n-1}||x^{n-1} - \cdots - ||A_1||x - ||A_0|| = 0$ and $||A_n||x^n + ||A_{n-1}||x^{n-1} + \cdots + ||A_1||x - ||A_0^{-1}||^{-1} = 0$, respectively, for any matrix norm. [1]

Improved Cauchy Theorem. Let $P(z)$ be defined as in (1), with A_n nonsingular. Denote by k the smallest positive integer such that A_{n-k} is not the null matrix, and define $Q^{(L)}(z) = (A_n z^k - A_{n-k})P(z)$ and $Q^{(R)}(z) = P(z)(A_n z^k - Z_{n-k})$. If $A_n A_{n-k} = A_{n-k} A_n$ and $||A_n^{-2}||^{-1} = ||A_n||\,||A_n^{-1}||^{-1}$, then the Cauchy radii of $Q^{(L)}$ and $Q^{(R)}$ are not larger than the Cauchy radius of P when the same matrix norm is used for all radii. [2]

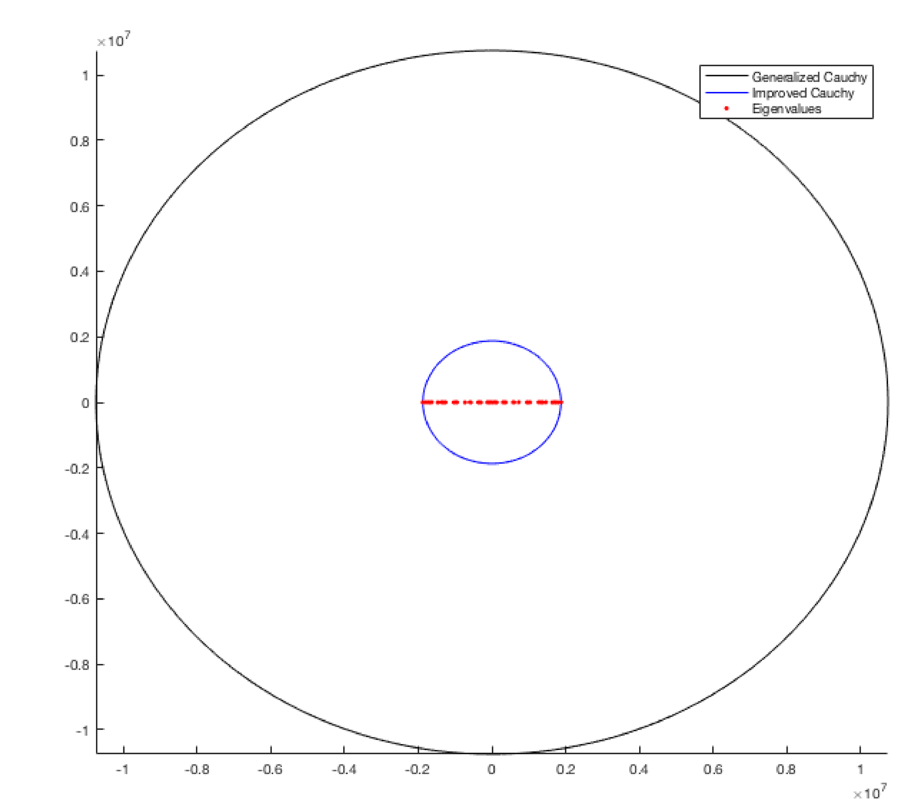

Figure 2: cd_player (above) improved by an order of magnitude.

In Figure 2 and Table 1, we compare the Generalized Cauchy Theorem with its improvement and observe significant improvement in many cases. Below are some of the most significant.

Problem	Cauchy	Improvement	Statistic
cd_player	1.07E+07	1.87E+06	**3.82E+07**
omnicam1	18.80	4.110	**5.469**
power_plant	9.94E+04	6.45E+03	**16.29**
speaker_box	8.01E+07	9.18E+04	**1.05E+03**

Table 1: Data from a few of the tested problems.

Pellet Bound Results

Generalized Pellet Theorem. Let $P(z)$ be defined as in (1) with $A_0 \neq 0$. Let A_ℓ be invertible for some ℓ with $1 \leq \ell \leq n-1$, and let the polynomial $||A_n||x^n + ||A_{n-1}||x^{n-1} + \cdots + ||A_{\ell+1}||x^{\ell+1} - ||A_\ell^{-1}||^{-1}x^\ell + ||A_{\ell-1}||x^{\ell-1} + \cdots + ||A_1||x + ||A_0||$ have two distinct positive roots ρ_1 and ρ_2 with $\rho_1 < \rho_2$ for any matrix norm. Then $\det(P)$ has exactly ℓm zeros in or on the disk $|z| = \rho_1$ and no zeros in the open annular ring $\rho_1 < |z| < \rho_2$. [1]

Improved Pellet Theorem. Let $P(z)$ be defined as in (1) with A_ℓ nonsingular, and with Pellet ℓ-radii ρ_1 and ρ_2, where $1 \leq \ell \leq n-1$ and $0 < \rho_1 < \rho_2$. Denote by k the smallest positive integer such that $A_{\ell-k}$ is not the null matrix, let $A_\ell A_{\ell-k} = A_{\ell-k} A_\ell$, and define $Q^{(L)}(z) = (A_\ell z^k - A_{\ell-k})P(z)$ and $Q^{(R)}(z) = P(z)(A_\ell z^k - Z_{\ell-k})$. If $||A_\ell^{-2}|| = ||A_\ell^{-1}||\,||A_\ell||^{-1}$, then $Q^{(L)}$ has Pellet $(\ell+k)$-radii $\sigma_1^{(L)}$ and $\sigma_2^{(L)}$, satisfying $0 < \sigma_1^{(L)} \leq \rho_1 < \rho_2 \leq \sigma_2^{(L)}$, and $\det(P)$ has exactly ℓm zeros in or on the circle $z = \sigma_1^{(L)}$, and no zeros in the open annular ring $\sigma_1^{(L)} < z < \sigma_2^{(L)}$. An analogous result holds for $Q^{(R)}$. [3]

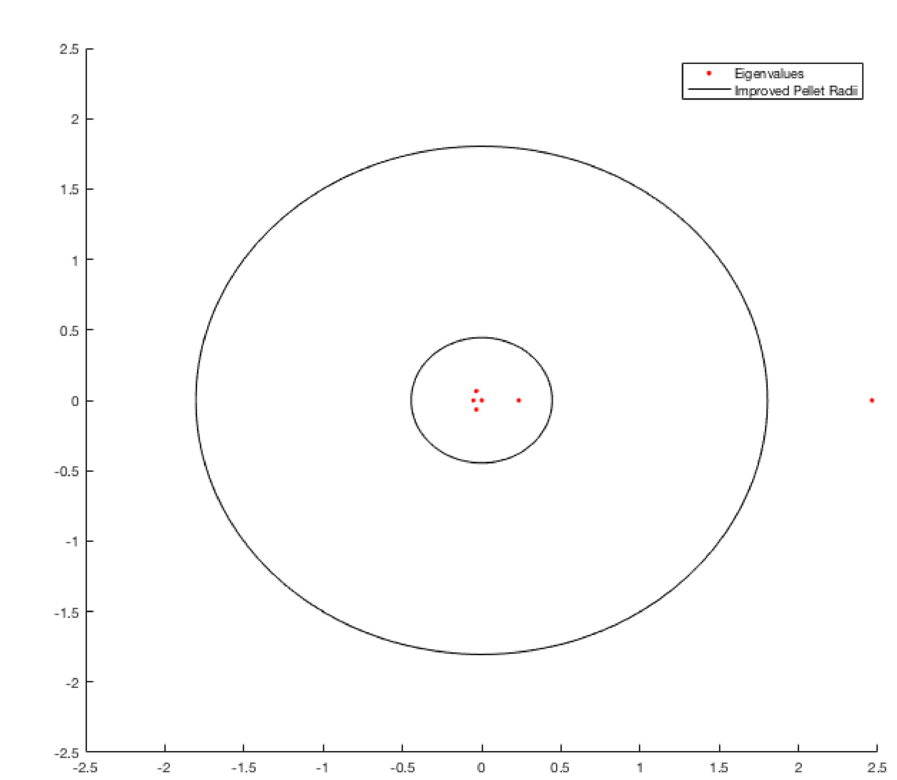

Figure 3: Improved Pellet radii for the bilby problem.

Analysis suggests that the improved Pellet radius is usually only slightly better than the Pellet radius. Nevertheless, the most interesting case is in the bilby problem (pictured in Figure 3). Here, the improved theorem returned a radius while the original theorem did not.

Conclusion

We surveyed improvements of Cauchy and Pellet radii for a large number of engineering problems, some shown here, proving their practical worth. Given their low computational cost, there is no reason not to apply the improvements. To our knowledge, this is the first such survey of real-world applications, previous comparisons having relied on artificial, randomly generated problems.

Further Research

Further research can be done on the iterative quality of these improvements, different multipliers with guaranteed improvement (currently nonexistent), and on left- and right-multiplication of the polynomial multipliers, which may affect sparsity, depending on the structure of the matrix coefficients.

References

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Acknowledgements

This research was carried out with a grant from the Kuehler undergraduate research program at Santa Clara University.

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