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Connective Stability of Discontinuous Dynamic Systems[1]

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Abstract. Connective stability is defined for interconnected systems under discontinuous structural perturbations. Stability conditions are established using Lipschitz and C^1-type Lyapunov functions. A considerable flexibility of the conditions is achieved by assuming the functions to be parameter dependent. For efficient testing, the obtained stability conditions are converted to convex optimization problems via the theory of M-matrices.

Key Words. Connective stability, discontinuous systems, Filippov solution, vector Lyapunov functions, polytopic uncertainty, M-matrices.

1. Introduction

The concept of connective stability was introduced to study interconnected systems subject to structural perturbations whereby the subsystems are disconnected and connected again during operation (Refs. 1–2). Interconnection matrices having coefficients that depend on time and state were used to model structural perturbations. While the coefficients were assumed to be piecewise-continuous in time, they were allowed to be only continuous in the state, thus ruling out abrupt changes of the system structure that depend on both time and state.

The objective of this paper is to present new results on the connective stability of discontinuous interconnected systems using the theory of

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differential equations with discontinuous right-hand side (Refs. 3–8). In particular, we will rely on the results obtained for discontinuous interconnected systems (Ref. 9), hybrid systems (Ref. 10), and the comparison principle for connective stability analysis (Ref. 11). To establish the connective stability conditions, we will consider two types of vector Lyapunov functions: regular Lipschitz functions and C^1 functions. Our analysis will follow the use of scalar Lyapunov functions proposed in Refs. 4–5 and 7–8. Since the choice of suitable vector Lyapunov functions for uncertain interconnected systems is a difficult issue, parameter-dependent functions are utilized (Refs. 12–13). Interconnection matrices are regarded as unknown perturbations that reside in a polytope, and stability testing is reduced to checking if a certain matrix polytope is a polytope of M-matrices. For testing purposes, three independent convex optimization problems are formulated, which involve only the vertices of the matrix polytope (Ref. 13).

2. Lyapunov Stability for Discontinuous Dynamic Systems

Let us consider a vector differential equation

$$\dot{x} = f(t, x), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state at time $t \in \boldsymbol{T} = [t_0, +\infty)$ and $f: \boldsymbol{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is a measurable function defined almost everywhere in an open or closed region $\boldsymbol{Q} \subseteq \boldsymbol{T} \times \mathbb{R}^n$ such that, for an arbitrary closed bounded region $\boldsymbol{D} \subseteq \boldsymbol{Q}$, there exists a summable function $\mu(t)$ satisfying $\|f(t, x)\| \leq \mu(t)$ almost everywhere in $\boldsymbol{D}$ (Condition B in Ref. 3). The absolutely continuous solutions of (1) in the sense of Filippov are denoted by $x(t; t_0, x_0)$ with $x(t_0) = x_0$ satisfying for almost all t the differential inclusion

$$\dot{x} \in K[f](t, x), \tag{2}$$

with

$$K[f](t, x) = \bigcap_{\delta > 0} \bigcap_{\mu\Omega = 0} \overline{\text{conv}}\, f(t, B(x, \delta) - \Omega), \tag{3}$$

where conv denotes convex hull, $\overline{\text{conv}}$ denotes convex closure, and $\bigcap_{\mu\Omega = 0}$ denotes the intersection over all sets Ω of measure zero. In other words, the set $K[f](t, x)$ is the smallest closed convex set containing all limit values of the function $f(t, x')$, where x', in tending to x, spans almost the whole neighborhood (that is, except on a set of measure zero) of the point x. Furthermore, we assume that $0 \in K[f](t, 0)$, for all $t \in \boldsymbol{T}$, and that $x = 0$ is the unique equilibrium point. To establish stability of the equilibrium, we will

use two types of Lyapunov functions: regular Lipschitz continuous Lyapunov functions and Lyapunov functions with the continuous first derivative, that is, functions which belong to C^1.

To consider a regular Lipschitz Lyapunov function, we need the following definitions and a lemma (Refs. 14–15).

Definition 2.1. Clarke's generalized derivative for a locally Lipschitz function $v: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\partial v(t, x) = \overline{\text{conv}}\{\lim(\text{grad}_{t,x}(v)) \mid (t_i, x_i) \to (t, x), (t_i, x_i) \notin \Omega_v\}, \tag{4}$$

where Ω_v is the set of measure zero where the gradient of v is not defined.

Definition 2.2. The generalized directional derivative is defined as

$$v^o(z; u) = \limsup_{w \to z,\, h \downarrow 0} [v(w + hu) - v(w)]/h. \tag{5}$$

Lemma 2.1. Let v be Lipschitz near z. Then,

$$v^o(z; u) = \max\{\langle \xi, u \rangle \mid \xi \in \partial v(z)\}. \tag{6}$$

Definition 2.3. The function v is regular at z provided that v is Lipschitz near z and admits the directional derivative $v'(z; u)$ for all u, with $v'(z; u) = v^o(z; u)$.

Remark 2.1. All functions that are continuously differentiable at z are regular at z. Also, convex functions which are Lipschitz near z are regular at z, that is, norm functions are regular.

Now, we are ready to state the following theorem which is a generalization of the standard Lyapunov stability theorem (e.g., Ref. 16) to differential equations with the discontinuous right-hand side when the Lyapunov function $v(t, x)$ is a regular Lipschitz function (Ref. 8).

Theorem 2.1. Let $v: \boldsymbol{T} \times \mathbb{R}^n \to \mathbb{R}_+$ be a regular Lipschitz function satisfying $v(t, 0) = 0$ and the inequalities

$$\phi_1(\|x\|) \leq v(t, x) \leq \phi_2(\|x\|), \tag{7a}$$

$$\dot{v}(t, x)_{(1)} \leq -\phi_3(\|x\|), \tag{7b}$$

for all $(t, x) \in Q$, and for some comparison functions $\phi_1, \phi_2, \phi_3 \in \mathscr{K}$, where $\tilde{v}(t, x)_{(1)}$ is defined as

$$\tilde{v}(t, x)_{(1)} = \bigcap_{\xi \in \partial v(t, x(t))} \xi^T \begin{bmatrix} 1 \\ K[f](t, x) \end{bmatrix}. \tag{8}$$

Then, the equilibrium $x = 0$ of (1) is uniformly asymptotically stable. If all the assumptions hold globally, that is, $Q = T \times \mathbb{R}^n$ and the comparison functions ϕ_1 and ϕ_2 are radially unbounded, that is, $\phi_1, \phi_2 \in \mathscr{K}_\infty$, the equilibrium $x = 0$ is uniformly asymptotically stable in the large.

Remark 2.2. We should note that the third inequality in (7) can be replaced by the following:

$$D^+ v(t, x(t))_{(1)} \leq -\phi_3(\|x\|), \qquad \text{a.e. in } T, \tag{9}$$

where $D^+ v(t, x(t))_{(1)}$ is the Dini derivative with respect to Eq. (1). Then, all the statements of Theorem 2.1 remain valid.

Remark 2.3. If the function $f(t, x)$ is a piecewise-continuous function, then we can replace inequality (7b) by the following (Ref. 4):

$$\begin{aligned} D^+ v(t, x(t))_{(1)} &= [D_h^+ v(t+h, x(t)+hy)]|_{h=0} \\ &\leq -\phi_3(\|x\|), \qquad \text{a.e. in } T, \end{aligned} \tag{10}$$

where $y = \dot{x} = f(t, x)$ and $[D_h^+ v(t+h, x(t)+hy)]|_{h=0}$ is the Dini derivative with respect to h evaluated at $h = 0$. This condition is easier to compute and takes care of the case when the solution may go along a line or a surface on which grad (v) does not exist. Again, all the statements of Theorem 2.1 remain valid.

Now, let us consider the next theorem where we assume that the Lyapunov function $v(t, x)$ has a continuous first derivative, that is, $v(t, x) \in C^1$. Then, the Lyapunov stability theorem has the following form (Ref. 4).

Theorem 2.2. Let $v: T \times \mathbb{R}^n \to \mathbb{R}_+$ be a C^1 function satisfying $v(t, 0) = 0$ and the inequalities

$$\phi_1(\|x\|) \leq v(t, x) \leq \phi_2(\|x\|), \tag{11a}$$

$$\sup_{y \in K[f](t, x)} (v_t + \text{grad}_x(v) y) \leq -\phi_3(\|x\|), \tag{11b}$$

for all $(t, x) \in Q$ and for some $\phi_1, \phi_2, \phi_3 \in \mathscr{K}$. Then, the equilibrium $x = 0$ is uniformly asymptotically stable. If all the assumptions hold globally, that is, $Q = T \times \mathbb{R}^n$ and the comparison functions ϕ_1 and ϕ_2 are radially unbounded, that is, $\phi_1, \phi_2 \in \mathscr{K}_\infty$, then the equilibrium $x = 0$ is uniformly asymptotically stable in the large.

Remark 2.4. If $f(t, x)$ is a piecewise-continuous function, then we can replace inequality (11b) with the inequality

$$v_t + \text{grad}_x(v) f(t, x) \leq -\phi_3(\|x\|), \tag{12}$$

that is satisfied only in the domains of continuity of the function $f(t, x)$; see Ref. 4. It is important to note that this statement is valid only if the Lyapunov function belongs to the class C^1.

Example 2.1. Let us reconsider a discontinuous system (Ref. 4),

$$\dot{x}_1 = 2 \operatorname{sign} x_1, \tag{13a}$$

$$\dot{x}_2 = -4 \operatorname{sign} x_2. \tag{13b}$$

If we choose a regular Lipschitz Lyapunov function $v(x) = |x_1| + |x_2|$, then

$$\tilde{v}_{(13)} = \dot{v}_{(13)} = -2 < 0 \tag{14}$$

in the domains of continuity, that is, for $x_1 x_2 \neq 0$. However, on the positive part of the x_1 axis, where the right-hand side of (13) is discontinuous, using (8) we obtain

$$\tilde{v}_{(13)} = (0, 1, [-1, 1]) \times (1, 2, [-4, 4])^T = [-2, 6], \tag{15}$$

which is clearly a set containing positive elements. Therefore, the conditions of Theorem 2.1 do not hold, since they are not valid on the line of discontinuity of (13). We can also use Remark 2.3 and compute

$$[D_h^+ v(t+h, x(t)+hy)]|_{h=0} = 2(\operatorname{sign} x_1(t))^2 - 4(\operatorname{sign} x_2(t))^2, \qquad \text{a.e. in } T, \tag{16}$$

which is also positive on the positive part of the x_1 axis.

By using a C^1 Lyapunov function $v(x) = x_1^2 + x_2^2$ and computing its derivative

$$\dot{v}_{(13)} = 2x_1 \operatorname{sign} x_1 - 4x_2 \operatorname{sign} x_2, \qquad x_1 x_2 \neq 0, \tag{17}$$

we notice that, in the sector described by $x_1 > x_2/2 > 0$, the Lyapunov function derivative is greater than zero. In other words, when we use a C^1 Lyapunov function, we need to check the sign of the derivative only in the domains of continuity, as opposed to the case when we use a Lipschitz Lyapunov function and in addition have to check the sign of the derivative on the lines of discontinuity.

3. Connective Stability

The concept of connective stability (Refs. 1–2 and 17–27) has been defined for interconnected systems. Its main objective is to capture the effect on stability of the essential uncertainty residing in the interconnections of complex systems.

In order to study connective stability, let us consider the system

$$\dot{x}_i = g_i(t, x_i) + h_i(t, x), \qquad i \in N, \tag{18}$$

which is an interconnection of N decoupled subsystems

$$\dot{x}_i = g_i(t, x_i), \qquad i \in N, \tag{19}$$

where $x_i \in \mathbb{R}^{n_i}$ is the state of the ith subsystem, the function g_i: $\boldsymbol{T} \times \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ satisfies Condition B and represents the dynamics of the ith subsystem, h_i: $\boldsymbol{T} \times \mathbb{R}^{n} \to \mathbb{R}^{n_i}$ is the interconnection between the ith subsystem and the rest of the system, and $N = \{1, 2, \ldots, N\}$. We assume that $0 \in K[g_i](t, 0)$ and $0 \in K[h_i](t, 0)$, for all $t \in \boldsymbol{T}$. The subsystems are not overlapping, so that

$$x = (x_1^T, \ldots, x_N^T)^T \quad \text{and} \quad n = \sum_{i=1}^{N} n_i.$$

To describe the sudden structural changes (either intentional or accidental) that the system undergoes during operation, we represent the interconnections as in Ref. 2,

$$h_i(t, x) = \sum_{j=1}^{N} e_{ij}(t, x) h_{ij}(t, x), \tag{20}$$

but in this paper e_{ij}: $\boldsymbol{T} \times \mathbb{R}^n \to \boldsymbol{E}$ are in general discontinuous functions and are elements of the $N \times N$ interconnection matrix $E = (e_{ij})$ belonging to a set $\boldsymbol{E} \subset \mathbb{R}^{N \times N}$. In order to define the set $\boldsymbol{E}$, we further limit the size of interconnections imposing the constraints

$$\|h_i(t, x)\| \leq \sum_{j=1}^{N} \bar{e}_{ij} \beta_{ij} \phi_{3j}(\|x_j\|), \quad \text{for all } (t, x) \in \boldsymbol{T} \times \mathbb{R}^n, \tag{21}$$

where the β_{ij}'s are nonnegative numbers and $\bar{e}_{ij}$'s are the elements of an $N \times N$ fundamental interconnection matrix $\bar{E} = (\bar{e}_{ij})$ residing in a polytope (Refs. 20–21)

$$\bar{\boldsymbol{E}} = \text{conv}\{\bar{E}^k\}, \tag{22}$$

which is a convex hull of constant $N \times N$ vertex matrices $\bar{E}^k > 0$, $k \in \boldsymbol{m} = \{1, 2, \ldots, m\}$. Now, we are ready to define set $\boldsymbol{E}$ as

$$\boldsymbol{E} = \{E: \exists\, \bar{E} \in \bar{\boldsymbol{E}}, |E| \leq \bar{E}\}. \tag{23}$$

In this way, we associate a polytope $\boldsymbol{E}$ with the system (18) and state a new definition of polytopic connective stability (Ref. 21).

Definition 3.1. A system (18) is connectively stable if it is stable in the sense of Lyapunov for all $E \in \boldsymbol{E}$.

It is important to note that, in Definition 3.1 and throughout the paper, we use a single Lyapunov function. It is well known (Ref. 28) that, if we have a piecewise-continuous system where Lyapunov stability is established in each domain of continuity with a different Lyapunov function, the overall system might be unstable.

4. Convex *M*-Matrices

Because of the crucial role of M-matrices (e.g., Refs. 29–30) in establishing our connective stability results we present in this section some of their properties. We denote the set of M-matrices as $\boldsymbol{M}_n$ and start with the set of matrices $A = (a_{ij})$ with nonpositive off-diagonal elements denoted as $\boldsymbol{Z}_n$. The following theorem provides a number of M-matrix properties.

Theorem 4.1. Let $A \in \boldsymbol{Z}_n$. Then, $A \in \boldsymbol{M}_n$ if and only if it satisfies one of the following equivalent conditions:

(i) There is a vector $x > 0$ such that $Ax > 0$.
(ii) There is a vector $x > 0$ such that $A^T x > 0$.
(iii) There is a positive diagonal matrix $D = \text{diag}\{d_1, \ldots, d_n\}$ such that the matrix $C = A^T D + DA$ is positive definite. The matrix D is called the positive diagonal Lyapunov solution.

Let us define a convex polytope $\boldsymbol{A} = \text{conv}\{A^k\}$, where A^k are the vertices of $\boldsymbol{A}$ and $k \in \boldsymbol{m}$. We want to present sufficient conditions in terms of A^k for $\boldsymbol{A}$ to be a polytope of M-matrices. This objective requires the convexity of M-matrices, which is known to be false in general. Horn and Johnson (Ref. 31) established the necessary and sufficient conditions for a convex combination of two M-matrices to be an M-matrix, but a generalization of their conditions to a polytope of matrices is an open problem.

The rowwise and columnwise dominance were introduced by Fan (Ref. 32) as sufficient conditions for a convex combination of pair of M-matrices to be an M-matrix. A generalization of the rowwise condition to matrix polytopes is the following result (Ref. 13).

Theorem 4.2. Let $A^k \in \boldsymbol{M}_n$ for all $k \in \boldsymbol{m}$. Define the augmented matrices $\bar{A}^k \in \mathbb{R}^{n\times(n+1)}$ such that $\bar{A}^k = [A^k, -e]$, where $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n_+$ and the augmented vector $\bar{x} = (x_1, x_2, \ldots, x_n, x_{n+1})^T \in \mathbb{R}^{n+1}$. Consider the following linear programming problem:

$$\max \quad x_{n+1}, \tag{24a}$$

$$\text{s.t.} \quad \bar{A}^k \bar{x} \geq 0, \quad k \in \boldsymbol{m}, \tag{24b}$$

$$\bar{x} \geq 0, \qquad x_{n+1} \leq 1. \tag{24c}$$

Then, $\boldsymbol{A}$ is a polytope of M-matrices if $x^0_{n+1} = \max x_{n+1} = 1$.

Remark 4.1. Notice that the problem (24) is convex; therefore, if there is a solution to (24) a linear program (e.g., simplex method) would find it. The idea of Theorem 4.2 was proposed by Burgat and Bernussou (Ref. 33) in the context of connective stability of large-scale systems under periodic structural perturbations.

The columnwise version of Theorem 4.2 (Ref. 13) is as follows.

Theorem 4.3. Let $A^k \in \boldsymbol{M}_n$ for all $k \in \boldsymbol{m}$. Define the augmented matrices $\tilde{A}^k \in \mathbb{R}^{n\times(n+1)}$ such that $\tilde{A}^k = [(A^k)^T, -e]$. From the linear programming problem (24), it follows that there exists a vector $\tilde{d} > 0$ such that $A^T(\alpha)\,\tilde{d} > 0$ if and only if $x^0_{n+1} = 1$. The condition $A^T(\alpha)\,\tilde{d} > 0$ implies that $\boldsymbol{A}$ is a polytope of M-matrices.

Another way to establish $\boldsymbol{A}$ as a polytope of M-matrices is to apply the concept of simultaneous Lyapunov functions introduced by Horisberger and Belanger (Ref. 34). Using condition (iii) of Theorem 4.1, we get the following theorem (Ref. 13).

Theorem 4.4. Let $A^k \in \boldsymbol{M}_n$ for all $k \in \boldsymbol{m}$. Then, the matrices A^k have a common positive diagonal Lyapunov solution D if the following system of linear matrix inequalities is feasible:

$$(A^k)^T D + D A^k \succ 0, \qquad k \in \boldsymbol{m}, \tag{25a}$$

$$D \succ 0, \tag{25b}$$

where $A \succ B$ denotes that $A - B$ is positive definite.

It is obvious that the existence of D such that inequalities (25) are satisfied implies that $\boldsymbol{A} \subset \boldsymbol{M}_n$. Theorem 4.4 is attractive because there are efficient algorithms for solving linear matrix inequalities (Ref. 35).

The fact that the conditions given by Theorems 4.2 to 4.4 are mutually independent has been shown in Ref. 13.

5. Lipschitz-Type Vector Lyapunov Functions

Connective stability analysis is carried out within the concept of vector Lyapunov functions (Refs. 36–37). The concept associates with a dynamic system several scalar functions in such a way that each of them determines a desired stability property in a part of the state space where the others fail to do so. In this section, we will consider Lipschitz-type vector Lyapunov functions.

Let us first assume that with each subsystem we associate a regular Lipschitz continuous function v_i: $\boldsymbol{T} \times \mathbb{R}^{n_i} \to \mathbb{R}_+$, which in addition satisfies a Lipschitz condition in x_i; that is, there is a number $k_i > 0$ such that

$$\|v_i(t, x_i') - v_i(t, x_i'')\| \leq k_i \|x_i' - x_i''\|, \tag{26}$$

for all $t \in \boldsymbol{T}$ and $x_i', x_i'' \in \mathbb{R}^{n_i}$. Furthermore, we require that $v_i(t, 0) = 0$ and that the function $v_i(t, x_i)$ satisfies the inequalities

$$\phi_{1i}(\|x_i\|) \leq v_i(t, x_i) \leq \phi_{2i}(\|x_i\|), \tag{27a}$$

$$\dot{v}_i(t, x_i)_{(19)} \leq -\phi_{3i}(\|x_i\|), \tag{27b}$$

for all (t, x) in $\boldsymbol{T} \times \mathbb{R}^{n_i}$ and for some $\phi_{1i}, \phi_{2i} \in \mathcal{K}_\infty$, $\phi_{3i} \in \mathcal{K}$, where

$$\dot{v}_i(t, x_i)_{(19)} = \bigcap_{\xi \in \partial v_i(t, x_i(t))} \xi^T \begin{bmatrix} 1 \\ K[g_i](t, x_i) \end{bmatrix} \tag{28}$$

and $\partial v_i(t, x_i(t))$ represents Clarke's generalized gradient. Now, the vector function v: $\boldsymbol{T} \times \mathbb{R}^n \to \mathbb{R}^N_+$, which is defined as $v = (v_1, v_2, \ldots, v_N)^T$, is a Lipschitz-type vector Lyapunov function.

We follow the standard procedure (Ref. 2) and define the $N \times N$ test matrix $W = (w_{ij})$ as

$$w_{ij} = \begin{cases} 1 - \bar{e}_{ii}\beta_{ii}k_i, & i = j, \\ -\bar{e}_{ij}\beta_{ij}k_i, & i \neq j. \end{cases} \tag{29}$$

The matrix W has nonpositive off-diagonal elements. This means that W is an M-matrix if and only if, for any element-by-element positive vector c, there is an element-by-element positive vector d such that $d^T W = c^T$. Since the dependence of W on $\bar{E}$ is of interest, we shall use the explicit notation $W(\bar{E})$, $c(\bar{E})$, $d(\bar{E})$.

Theorem 5.1. The equilibrium $x = 0$ of the polytopic discontinuous interconnected system (18) is connectively stable if the matrix $W(\bar{E})$ is an M-matrix for all $\bar{E} \in \bar{\boldsymbol{E}}$.

Proof. Since $v_i(\cdot)$ is a Lipschitz function and $(t, x_i(t))$ is absolutely continuous function, it follows that $v_i(t, x_i(t))$ is absolutely continuous function of t and that $D^+v_i(t, x_i)$ exists almost everywhere. At a point where $x_i(t)$ and $v_i(t, x_i(t))$ are both differentiable (this is true almost everywhere), we compute

$$\begin{aligned}
&D^+v_i(t, x_i)_{(18)}\\
&= \limsup_{h\downarrow 0} [v_i(t+h, x_i(t+h)) - v_i(t, x_i(t))]/h\\
&= \limsup_{h\downarrow 0} [v_i(t+h, x_i(t) + h[g_i(t, x_i) + h_i(t, x)]) + o(h) - v_i(t, x_i(t))]/h\\
&\le \limsup_{h\downarrow 0} [v_i(t+h, x_i(t) + hg_i(t, x_i)) - v_i(t, x_i(t))]/h\\
&+ \limsup_{h\downarrow 0} [v_i(t+h, x_i(t) + hg_i(t, x_i) + hh_i(t, x)) - v_i(t+h, x_i(t) + hg_i(t, x_i))]/h\\
&\le \bar{v}_i(t, x_i)_{(21)} + k_i\|h_i\|\\
&\le -\phi_{3i}(\|x_i\|) + k_i \sum_{j=1}^{N} \bar{e}_{ij}\beta_{ij}\phi_{3j}(\|x_j\|)\\
&\le -\sum_{j=1}^{N} \bar{w}_{ij}\phi_{3j}(\|x_j\|), \qquad \text{a.e. in } \boldsymbol{T}.
\end{aligned} \tag{30}$$

Let us consider the function

$$\upsilon(t, x, \bar{E}) = d^T(\bar{E})v(t, x) \tag{31}$$

as a parameter-dependent Lyapunov function (Ref. 12) for the discontinuous polytopic system (18), where $d(\bar{E})$ is a positive vector which depends on $\bar{E}$ and whose existence has yet to be established for each $\bar{E}\in\boldsymbol{E}$. We note that the function $\upsilon(t, x, \bar{E})$ is a regular Lipschitz function, since from its definition it is equal to a finite nonnegative linear combination of regular Lipschitz functions (Ref. 15), and function v: $\boldsymbol{T}\times\mathbb{R}^n\to\mathbf{R}_+^N$ is vector Lyapunov function. Using (30)–(31), we get

$$D^+\upsilon(t, x, \bar{E})_{(18)} \le -c^T(\bar{E})\phi_3(\|x\|), \qquad \text{a.e. in } \boldsymbol{T}, \tag{32}$$

where $\phi_3 = (\phi_{31}, \phi_{32}, \dots, \phi_{3N})^T$ and $d^T(\bar{E})W(\bar{E}) = c^T(\bar{E})$.

Now, since $W(\bar{E})$ is an M-matrix for every $\bar{E}\in\hat{\boldsymbol{E}}$, it follows that there is a positive vector $d(\bar{E})$ whenever $c(\bar{E})$ is a positive vector (Theorem 4.1). This function implies that, for each $\bar{E}\in\boldsymbol{E}$, we have

$$\phi_I(\|x\|) \le \upsilon(t, x, \bar{E}) \le \phi_{II}(\|x\|), \tag{33a}$$

$$D^+\upsilon(t, x, \bar{E})_{(18)} \le -\phi_{III}(\|x\|), \qquad \text{a.e. in } \boldsymbol{T}, \tag{33b}$$

where the $\mathscr{K}$-functions are

$$\phi_I(\|x\|) = \sum_{i=1}^{N} d_i(\bar{E})\phi_{1i}(\|x_i\|), \tag{34a}$$

$$\phi_{II}(\|x\|) = \sum_{i=1}^{N} d_i(\bar{E})\phi_{2i}(\|x_i\|), \tag{34b}$$

$$\phi_{III}(\|x\|) = \sum_{i=1}^{N} c_i(\bar{E})\phi_{3i}(\|x_i\|). \tag{34c}$$

From Eqs. (33)–(34) and by Remark 2.2, we conclude the connective stability of the equilibrium $x = 0$ of discontinuous system (18). □

Remark 5.1. If the right-hand side of (18) is a piecewise-continuous function, then by following Filippov (Ref. 4) and the derivation in (30) we can easily show that

$$\begin{aligned}
D^+v_i(t, x_i(t))_{(18)} &= [D_h^+ v_i(t+h, x_i(t) + hy)]|_{h=0}\\
&\le [D_h^+ v_i(t+h, x_i(t) + hy_1)]|_{h=0} + k_i \sum_{j=1}^{N} \bar{e}_{ij}\beta_{ij}\phi_{3j}(\|x_j\|)\\
&\le -\sum_{j=1}^{N} \bar{w}_{ij}\phi_{3j}(\|x_j\|), \qquad \text{a.e. in } \boldsymbol{T},
\end{aligned} \tag{35}$$

where $y = f_i(t, x)$ and $y_1 = g_i(t, x_i)$. Now, instead of inequality (27b), we assume that the following condition is satisfied:

$$[D_h^+ v_i(t+h, x_i(t) + hy_1)]|_{h=0} \le \phi_{3i}(\|x_i(t)\|), \qquad \text{a.e. in } \boldsymbol{T}, \tag{36}$$

for the solutions of (18) in the sense of Filippov while Theorem 5.1 remains valid. Again, the alternative inequality (36) might be more suitable for checking stability, since it does not involve calculations over sets.

6. C^1-Type Vector Lyapunov Functions

Let us now assume that, with each subsystem, we associate a scalar function v_i: $\boldsymbol{T}\times\mathbf{R}^{n_i}\to\mathbf{R}_+$, which is a C^1 function. We assume also that $v_i(t, 0) = 0$ and that the function $v_i(t, x_i)$ satisfies the inequalities

$$\phi_{1i}(\|x_i\|) \le v_i(t, x_i) \le \phi_{2i}(\|x_i\|), \tag{37a}$$

$$\sup_{y_i\in K[g_i](t, x_i)} [\partial v_i/\partial t + \text{grad}_{x_i}(v_i)y_i] \le -\phi_{3i}(\|x_i\|), \tag{37b}$$

for all (t, x) in $T \times \mathbb{R}^n$ and for some $\phi_{1i}, \phi_{2i} \in \mathcal{K}_\infty$, $\phi_{3i} \in \mathcal{K}$. Again, the vector function $v: T \times \mathbb{R}^n \to \mathbb{R}_+^N$, which is defined as $v = (v_1, v_2, \dots, v_N)^T$ is a C^1-type vector Lyapunov function.

In order to establish connective stability using C^1 functions, we impose the following restrictions on the interconnections:

$$\sup_{y_i \in K[h_i](t,x)} (\mathrm{grad}_{x_i}(v_i) y_i) \le \phi_{3i}^{1/2}(\|x_i\|) \sum_{j=1}^{N} \bar{e}_{ij} \xi_{ij} \phi_{3j}^{1/2}(\|x_j\|). \tag{38}$$

Now, following Araki (Ref. 38), we define the $N \times N$ test matrix $\hat{W} = (\hat{w})$ as

$$\hat{w}_{ij} = \begin{cases} 1 - \bar{e}_{ii}\xi_{ii}, & i = j, \\ -\bar{e}_{ij}\xi_{ij}, & i \ne j, \end{cases} \tag{39}$$

and prove the following theorem.

Theorem 6.1. The equilibrium $x = 0$ of the polytopic discontinuous interconnected system (18) is connectively stable if the matrix $\hat{W}(\bar{E})$ is an M-matrix for all $\bar{E} \in \bar{\mathbf{E}}$.

Proof. Let us select the function $\upsilon: T \times \mathbb{R}^n \to \mathbb{R}_+$ defined as

$$\upsilon(t, x, \bar{E}) = d^T(\bar{E}) v(t, x), \tag{40}$$

where the function $v: T \times \mathbb{R}^n \to \mathbb{R}_+^N$ is a vector Lyapunov function $v = (v_1, v_2, \dots, v_N)^T$, and where the existence of vector $d(\bar{E})$ is yet to be established. Since from Eq. (40) $v(t, x, \bar{E})$ is equal to a finite linear combination of C^1 functions, it follows that $v(t, x, \bar{E})$ is also in C^1. Then, we compute

$$\sup_{y \in K[f](t,x)} [\partial \upsilon/\partial t + \mathrm{grad}_x(\upsilon) y]$$

$$= \sup_{\substack{y_1 \in K[f_1](t,x) \\ \vdots \\ y_N \in K[f_N](t,x)}} \left[\sum_{i=1}^{N} d_i(\partial v_i/\partial t) + \sum_{i=1}^{N} d_i \,\mathrm{grad}_{x_i}(v_i) y_i \right]$$

$$\le \sup_{\substack{y_1 \in K[g_1](t,x_1) + K[h_1](t,x) \\ \vdots \\ y_N \in K[g_N](t,x_N) + K[h_N](t,x)}} \left[\sum_{i=1}^{N} d_i(\partial v_i/\partial t) + \sum_{i=1}^{N} d_i \,\mathrm{grad}_{x_i}(v_i) y_i \right]$$

$$\le \sup_{\substack{y_1 \in K[g_1](t,x_1) \\ \vdots \\ y_N \in K[g_N](t,x_N)}} \left[\sum_{i=1}^{N} d_i(\partial v_i/\partial t) + \sum_{i=1}^{N} d_i \,\mathrm{grad}_{x_i}(v_i) y_i \right]$$

$$+ \sup_{\substack{y_1 \in K[h_1](t,x) \\ \vdots \\ y_N \in K[h_N](t,x)}} \left[\sum_{i=1}^{N} d_i \,\mathrm{grad}_{x_i}(v_i) y_i \right]$$

$$\le \sum_{i=1}^{N} d_i \left\{ \sup_{\substack{y_1 \in K[g_1](t,x_1) \\ \vdots \\ y_N \in K[g_N](t,x_N)}} [\partial v_i/\partial t + \mathrm{grad}_{x_i}(v_i) y_i] \right\}$$

$$+ \sum_{i=1}^{N} d_i \left[\sup_{\substack{y_1 \in K[h_1](t,x) \\ \vdots \\ y_N \in K[h_N](t,x)}} (\mathrm{grad}_{x_i}(v_i) y_i) \right]$$

$$\le \sum_{i=1}^{N} d_i \{ -\phi_{3i}(\|x_i\|) + \phi_{3i}^{1/2}(\|x_i\|) \sum_{j=1}^{N} \bar{e}_{ij} \xi_{ij} \phi_{3j}^{1/2}(\|x_j\|) \}$$

$$\le -(1/2) \phi_3^T(\|x_i\|) \{ \hat{W} D + D \hat{W} \} \phi_3^T(\|x_i\|), \tag{41}$$

where $\phi_3(\|x\|) = [\phi_{31}^{1/2}(\|x_1\|), \dots, \phi_{3N}^{1/2}(\|x_N\|)]^T$, $D = \mathrm{diag}\{d_1, \dots, d_N\}$. In the sequence of inequalities (41), we used the following two results (Ref. 7):

(i) Assume that $f_1, f_2: T \times \mathbf{R}^n \to \mathbf{R}^n$ are locally bounded; then,

$$K[f_1 + f_2](t, x) \subset K[f_1](t, x) + K[f_2](t, x). \tag{42}$$

(ii) Assume that $f_i: T \times \mathbf{R}^n \to \mathbf{R}^n$, $i \in N$ are locally bounded; then,

$$K[(f_1, \dots, f_N)^T](t, x) = (K[f_1](t, x), \dots, K[f_N](t, x))^T. \tag{43}$$

Now, we can write

$$\phi_I(\|x\|) \le \upsilon(t, x, \bar{E}) \le \phi_{II}(\|x\|), \tag{44a}$$

$$\sup_{y \in K[f](t,x)} [\partial \upsilon/\partial t + \mathrm{grad}_x(\upsilon) y] \le -\phi_{III}(\|x\|), \quad \text{in } T \times \mathbf{R}^n, \tag{44b}$$

where the $\mathcal{K}$-functions are

$$\phi_I(\|x\|) = \sum_{i=1}^{N} d_i(\bar{E}) \phi_{1i}(\|x_i\|), \tag{45a}$$

$$\phi_{II}(\|x\|) = \sum_{i=1}^{N} d_i(\bar{E}) \phi_{2i}(\|x_i\|), \tag{45b}$$

$$\phi_{III}(\|x\|) = (1/2) \phi_3^T(\|x\|) C(\bar{E}) \phi_3^T(\|x\|), \tag{45c}$$

since there exists a matrix D such that the matrix

$$C(\bar{E}) = \hat{W}^T(\bar{E}) D + D \hat{W}(\bar{E}) \tag{46}$$

is positive definite for any $\bar{E} \in \bar{\mathbf{E}}$.

From Eqs. (44)–(46) and by Theorem 2.2, it follows that the equilibrium $x = 0$ of the system (18) is connectively stable. □

Remark 6.1. It is important to note that the M-matrix conditions of Theorems 5.1 and 6.1 can be tested using convex programming tools via Theorems 4.2 to 4.4.

Remark 6.2. As we said earlier in the paper, when we have a C^1 Lyapunov function and the right-hand side of (18) is a piecewise-continuous function, it is sufficient for stability if we check derivative of the Lyapunov function only in the domains of continuity. Therefore, in this case, the analysis turns out to be very similar to the analysis for continuous dynamic systems.

7. Conclusions

We have formulated connective stability conditions for discontinuous interconnected systems in terms of both Lipschitz and C^1-type vector Lyapunov functions. Vector Lyapunov functions are parameter dependent due to the polytopic structure of the interconnection terms regarded as perturbations of the subsystem dynamics. Connective stability conditions are formulated as convex optimization problems relying on M-matrix theory. Therefore, if the problem is feasible, convex programming theory guarantees that a solution will be computed.

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