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Brief paper

# Control design with arbitrary information structure constraints ☆

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ABSTRACT

In this paper, a new method is proposed for designing robust control laws that are subject to arbitrary information structure constraints. The computation of the gain matrix is formulated in terms of a static output feedback problem, which can be efficiently solved using linear matrix inequalities. The resulting control laws ensure stability with respect to a broad class of additive nonlinear uncertainties in the system.

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## 1. Introduction

In the analysis of complex dynamic systems, it is quite common to encounter constraints on the information that is available for control purposes. More often than not, these constraints can be incorporated into the design by choosing an appropriate structure for the gain matrix $K$. An obvious example of such a situation arises in the context of decentralized control, where only local measurements are available. In this case, the information constraints can be taken into account directly by restricting matrix $K$ to a block-diagonal form (see *e.g.* Šiljak (1991) and the references therein). A similar approach can be applied to overlapping and bordered block-diagonal (BBD) control laws, since each of these structures can be uniquely associated with a specific pattern of information exchange between the subsystems (Bakule & Rodellar, 1995; Iftar, 1993; Ikeda & Šiljak, 1986; Sezer & Šiljak, 1991; Šiljak & Zečević, 2005; Zečević & Šiljak, 2005).

The special control structures noted above have received considerable attention in the past, and the body of literature on this subject is truly enormous. In contrast, only a few papers have been devoted to gain matrices with *arbitrary* nonzero patterns (Konstantinov, Patarinski, Petkov, and Khristov (1977), Sezer and Šiljak (1981), Šiljak (1991) and Wenk and Knapp (1980) are the only pertinent references that we could find on this topic). This is not entirely unexpected, given the generality of the problem and the related mathematical complications. We should point out, however, that irregular control structures have become increasingly relevant in recent years, due to the emergence of wireless networks and the commercial availability of communication satellites. These and other developments have expanded the possibilities for information exchange in the system, and have provided the designer with a great deal of additional flexibility. The potential benefits promise to be significant, and control schemes based on these technological advances are already being explored in a variety of engineering disciplines. A typical example of this trend is the ongoing research in electric power systems, which attempts to utilize remote information for improving the overall system performance (Kamwa, Grondin, & Hébert, 2001; Karlsson, Hemmingsson, & Lindahl, 2004).

In analyzing applications of this type, we should keep in mind that most practical systems permit only a limited range of communication patterns. Such restrictions are often the result of physical factors (such as large distances between certain subsystems), although economic and security considerations can play an important role as well. The problem of incorporating these structural constraints into control design has received considerable attention in the recent literature (*e.g.* Hristu and Morgansen (1999), Narendra, Oleng, and Mukhopadhyay (2006) and Walsh, Beldiman, and Bushnell (2001)). There has also been a concerted effort to understand how communication delays, noise and limited channel capacity impact the stability of the closed-loop system (for more details on this aspect of the problem, see Elia and Mitter (2001) or Liberzon and Hespanha (2005)).

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In this paper we will assume that the communication channels are ideal, and will focus exclusively on *structural constraints* that limit the flow of information in the system. Our primary objective in this context will be to propose an algorithm for designing robust control laws that are subject to preassigned nonzero patterns in the gain matrix (these patterns can be highly irregular in general). In Section 2 we show how such problems can be efficiently solved in the framework of convex optimization and linear matrix inequalities (*e.g.* Boyd, El Ghaoui, Feron, and Balakrishnan (1994), El Ghaoui and Niculescu (2000), Geromel, Bernussou, and Peres (1994) and Geromel, Bernussou, and de Oliveira (1999)). Some additional issues related to preconditioning are discussed in Section 3, which focuses on the case where matrix $A$ is unstable.

## 2. The design algorithm

Let us consider a nonlinear system of the form

$$\dot{x} = Ax + h(x) + Bu \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ represents the input and $A$ and $B$ are constant matrices of dimension $n \times n$ and $n \times m$, respectively. The nonlinearity $h(x)$ can be uncertain, but is assumed to satisfy the bound

$$h^{\mathrm{T}}(x)\, h(x) \leq \alpha^2 x^{\mathrm{T}} H^{\mathrm{T}} H x \tag{2}$$

where $H$ is a constant matrix and $\alpha$ is a scalar parameter. This parameter reflects the robustness of the system, and can be maximized by an appropriate choice of feedback.

Given a linear control law

$$u = Kx, \tag{3}$$

the global asymptotic stability of the closed-loop system can be established using a Lyapunov function

$$V(x) = x^{\mathrm{T}} P x, \tag{4}$$

where $P$ is a symmetric positive definite matrix. Defining $Y = \tau P^{-1}$ (where $\tau$ is a positive scalar), $L = KY$ and $\gamma = 1/\alpha^2$, the control design can be formulated as an LMI problem in $\gamma, \kappa_Y, \kappa_L, Y$ and $L$ (Šiljak & Stipanović, 2000; Zečević, Nešković, & Šiljak, 2004).

**Problem 1.** Minimize $a_1\gamma + a_2\kappa_Y + a_3\kappa_L$ subject to

$$Y > 0 \tag{5}$$

$$\begin{bmatrix} AY + YA^{\mathrm{T}} + BL + L^{\mathrm{T}}B^{\mathrm{T}} & I & YH^{\mathrm{T}} \\ I & -I & 0 \\ HY & 0 & -\gamma I \end{bmatrix} < 0 \tag{6}$$

$$\gamma - 1/\bar{\alpha}^2 < 0 \tag{7}$$

and

$$\begin{bmatrix} -\kappa_L I & L^{\mathrm{T}} \\ L & -I \end{bmatrix} < 0; \qquad \begin{bmatrix} Y & I \\ I & \kappa_Y I \end{bmatrix} > 0. \tag{8}$$

If the LMI optimization is feasible, the feedback law

$$u = LY^{-1}x \equiv Kx \tag{9}$$

is guaranteed to stabilize the closed-loop system for *all* nonlinearities that satisfy (2). Condition (7) ensures that $\alpha$ is greater than some prescribed value $\bar{\alpha}$, and the two inequalities in (8) bound the norm of the gain matrix as $\|K\| \leq \sqrt{\kappa_L \kappa_Y}$.We should also note in this context that Problem 1 maximizes a linear combination of variables $\gamma$, $\kappa_Y$ and $\kappa_L$. From that perspective, it is appropriate to view the obtained solution as a compromise between maximizing the robustness bound $\alpha$ and minimizing the norm of the gain matrix. The weighting coefficients in the cost function reflect the relative importance of the three terms, and are typically chosen as $a_1 = 0.01$, $a_2 = 10$ and $a_3 = 0.01$.

It is important to recognize that the approach outlined above places certain implicit structural limitations on matrix $K$. Indeed, although the LMI procedure allows us to assign an arbitrary nonzero pattern to matrix $L$, this pattern need not be preserved after multiplication by $Y^{-1}$. The exception, of course, is the case when $Y$ is chosen to be a diagonal matrix, but this tends to be a restrictive requirement which often leads to infeasibility. With that in mind, it is fair to say that optimization Problem 1 can realistically produce only certain special structures for $K$, such as block-diagonal or bordered block-diagonal (BBD) forms (*e.g.* Šiljak and Zečević (2005)).

In order to extend this approach to control laws with general structural constraints, let us consider a matrix $K$ whose nonzero pattern is *arbitrary*. We can always represent such a matrix in the form

$$K = K_0 C \tag{10}$$

where $K_0$ contains all the nonzero columns of $K$, and $C$ consists of the corresponding rows of the identity matrix. The LMI optimization in Problem 1 can be adapted to include problems of this type by virtue of the following lemma.

**Lemma 1.** *Let us assume that Problem 1 is feasible with matrices $Y$ and $L$ of the form*

$$\begin{aligned} Y &= \rho Y_0 + Q Y_Q Q^{\mathrm{T}} \\ L &= L_C U^{\mathrm{T}} \end{aligned} \tag{11}$$

*where $\rho$, $Y_Q$ and $L_C$ are* LMI *variables ($\rho > 0$ is a scalar parameter). Suppose further that $Y_0$, $Q$ and $U$ are constant matrices that satisfy*

$$U = Y_0 C^{\mathrm{T}} \tag{12}$$

*and*

$$Q^{\mathrm{T}} C^{\mathrm{T}} = 0. \tag{13}$$

*Then, the feedback law*

$$u = K_0 C x \tag{14}$$

*with $K_0 = \rho^{-1} L_C$ stabilizes system (1) for all nonlinearities that conform to bound (2).*

**Proof.** Conditions (11)–(13) ensure that

$$YC^{\mathrm{T}} = \rho Y_0 C^{\mathrm{T}} + Q Y_Q Q^{\mathrm{T}} C^{\mathrm{T}} = \rho U \tag{15}$$

and therefore

$$U^{\mathrm{T}} Y^{-1} = \rho^{-1} C \tag{16}$$

as well. Since Problem 1 is assumed to be feasible under these circumstances, it will produce a gain matrix of the form

$$K = LY^{-1} = L_C U^{\mathrm{T}} Y^{-1} = K_0 C \tag{17}$$

with $K_0 = \rho^{-1} L_C$. □

An immediate advantage of this result stems from the fact that matrices $K_0$ and $L_C$ have *identical* nonzero patterns. Consequently, in solving Problem 1 we can assign an *arbitrary* structure to $L_C$, and guarantee that it will be preserved in $K_0$. The following corollary further simplifies the optimization, and allows us to replace condition (8) with a *single* inequality.

**Corollary 1.** *Let $\mu > 0$ be a given positive number. Then, inequality*

$$\begin{bmatrix} -\rho\mu I & L_C^{\mathrm{T}} \\ L_C & -\rho\mu I \end{bmatrix} < 0 \tag{18}$$

*ensures that $\|K_0\| < \mu$.*

**Proof.** Let us first observe that matrix

$$M = \begin{bmatrix} I & (1/\rho\mu)L_C^{\mathrm{T}} \\ 0 & I \end{bmatrix} \tag{19}$$

is nonsingular by construction. If expression (18) is multiplied on the left by $M^{\mathrm{T}}$ and on the right by $M$, we obtain

$$\begin{bmatrix} -\rho\mu I & 0 \\ 0 & (1/\rho\mu)L_C L_C^{\mathrm{T}} - \rho\mu I \end{bmatrix} < 0 \tag{20}$$

which is equivalent to

$$L_C L_C^{\mathrm{T}} < \rho^2\mu^2 I \tag{21}$$

(since $\rho > 0$). Recalling that $\|K_0\|_2 = \left\|\rho^{-1}L_C\right\|_2$, we directly obtain $\|K_0\|_2 < \mu$. □

In concluding this section, we should note that matrix $C$ in (10) has dimension $p \times n$, where $p$ denotes the number of nonzero columns in $K$. As a result, any matrix $Q$ that satisfies (13) must have rank $\leq n - p$. If $n = p$, the term $QY_QQ^{\mathrm{T}}$ in (11) will vanish, which can adversely affect the feasibility of the LMI optimization. In order to circumvent this problem, we can always partition matrix $K$ as

$$K = K_1C_1 + K_2C_2 \tag{22}$$

where $K_1$ and $K_2$ consist of the even and odd columns of $K$, respectively, while $C_1$ and $C_2$ represent the appropriate rows of the identity matrix. Under such circumstances, the corresponding matrices $Q_1$ and $Q_2$ that satisfy (13) are guaranteed to have rank no higher than $(n + 1)/2$. An adjustment of this type requires a procedure in which Problem 1 is solved *twice*. In the first pass, we use matrix $A$ in inequality (6) and assume that the gain has the form $K = K_1C_1$. The algorithm is then repeated with $\tilde{A} = A + BK_1C_1$, in which case the gain matrix is assumed to be of the form $K = K_2C_2$.

**Remark 1.** The decomposition described in (22) is heuristic, and does not guarantee that the system will be stabilized with $K = K_1C_1$ (or $K = K_2C_2$, for that matter). Furthermore, even if Problem 1 is feasible with such a partitioning, the obtained solution may still be suboptimal. In view of that, it is usually advisable to consider several different partitionings of the columns of $K$, and select the one that produces the best results.

## 3. Preconditioning and related issues

Although Lemma 1 places no explicit constraints on the properties of matrix $Y_0$, in practice it is important to select it in a way that is consistent with inequalities (5) and (6). Without such a preconditioning the LMI optimization described in the previous section could easily become infeasible, since variables $\rho$, $Y_Q$ and $L_C$ cannot always offset a poor choice of $Y_0$.

When $A$ is a stable matrix the problem can be resolved rather easily, since $Y_0$ can be computed as the unique solution of the Lyapunov equation

$$AY_0 + Y_0A^{\mathrm{T}} = -I. \tag{23}$$

To see why such a choice is appropriate, it suffices to recall that $Y_0$ appears in inequalities (5) and (6) (since it is an additive component of matrix $Y$). The fact that $Y_0$ satisfies (23) and is positive definite is clearly conducive to the feasibility of Problem 1.

The identification of an appropriate $Y_0$ becomes considerably more complicated when $A$ is an *unstable* matrix. A natural approach in this case would be to *precondition* the optimization by computing a gain matrix $\tilde{K}$ that stabilizes $A$ (while conforming to the given nonzero pattern). Once such a matrix is obtained, Problem 1 can be solved using $\tilde{A} = A + B\tilde{K}$ instead of $A$, and $Y_0$ can be determined as the solution of the modified Lyapunov equation

$$\tilde{A}Y_0 + Y_0\tilde{A}^{\mathrm{T}} = -I. \tag{24}$$

In the following, we will explore this possibility, and identify several generic scenarios where such a preconditioning can be successfully performed.

**Case 1.** The simplest way to deal with the instability of $A$ is based on the following lemma.

**Lemma 2.** *Let $\hat{A} = A - \beta I$ be a stable matrix, and let $Y_0$ be the solution of Lyapunov equation*

$$(A - \beta I)Y_0 + Y_0^{\mathrm{T}}(A - \beta I)^{\mathrm{T}} = -I. \tag{25}$$

*Suppose also that the conditions of Lemma 1 are satisfied when $A$ is replaced by $\hat{A}$ and $H$ is set to equal the identity matrix. If Problem 1 produces a gain matrix $\tilde{K}$ and an $\alpha$ that satisfies $\alpha > \beta$, the resulting closed-loop matrix $\tilde{A} = A + B\tilde{K}$ is guaranteed to be stable. Furthermore, $\tilde{K}$ can be chosen to conform to an arbitrary nonzero pattern.*

**Proof.** Since the conditions of Lemma 1 are satisfied when $A$ is replaced by $\hat{A}$ and $H = I$, it follows that the closed-loop system

$$\dot{x} = (\hat{A} + B\tilde{K})x + h(x) \tag{26}$$

will be stable for all nonlinearities that satisfy

$$h^{\mathrm{T}}(x)\, h(x) \leq \alpha^2 x^{\mathrm{T}}x. \tag{27}$$

Recalling that $\alpha > \beta$ by assumption, inequality (27) must hold for $h(x) = \beta x$ as well. In this particular case, we have $\hat{A}x + h(x) = Ax$, which implies that $\tilde{A} = A + B\tilde{K}$ must be a stable matrix. Lemma 1 also ensures that $\tilde{K}$ has the form $\tilde{K} = \rho^{-1}L_CC$, which can accommodate an arbitrary nonzero pattern. □

**Remark 2.** Since the matrix $Y_0$ in (24) depends on $\tilde{A}$, it is advisable to iteratively solve Problem 1, using $\tilde{A}_i = A + B\tilde{K}_i$ to obtain $\tilde{K}_{i+1}$ $(i = 1, 2, \ldots)$. By doing so, it is possible to successively increase the value of the robustness bound $\alpha$. In principle, the iterations can continue until a specified upper bound for $\|K\|$ is reached.

**Case 2.** If the above strategy fails, it is necessary to consider more elaborate ways for preconditioning matrix $A$. One obvious possibility corresponds to the case when the system is *input decentralized* (e.g. Šiljak (1991)), and the gain matrix has a structure of the form

$$K = K_D + K_C \tag{28}$$

where $K_D = \mathrm{diag}\{K_{11}, \ldots, K_{NN}\}$ consists of *full* diagonal blocks, and $K_C$ has an arbitrary nonzero pattern. This type of situation arises in the context of interconnected subsystems which can exchange limited amounts of information through a fixed set of communication channels. The basic idea in this case would be to design *two* levels of control, the first of which uses local measurements and a decentralized feedback law $u = K_Dx$ to stabilize matrix $A$ (an LMI-based procedure for this type of design is described in Šiljak and Stipanović (2000)). The second level of control (which corresponds to $K_C$) would then enhance the system performance by exploiting the additional information that is exchanged between designated subsystems. The pattern of this exchange can be arbitrary, and is determined by the available communication channels. The gain matrix $K_C$ can be directly obtained by replacing matrix $A$ in Problem 1 with $\tilde{A} = A + B_D K_D$.

**Case 3.** The most general scenario corresponds to the case when matrix $B_D$ consists of diagonal blocks of dimension $n_i \times 1$ ($i = 1, 2, \ldots, N$), and $K$ is allowed to have an arbitrary nonzero pattern. Given the structure of $B_D$, it is natural to decompose $K$ in the form (28), where $K_D = \text{diag}\{K_{11}, \ldots, K_{NN}\}$ consists of diagonal blocks of $K$ whose dimensions are $1 \times n_i$ ($i = 1, 2, \ldots, N$), while $K_C$ represents the remainder of the matrix. We can then utilize $K_D$ to stabilize matrix $A$, and compute $K_C$ by solving Problem 1 with $\tilde{A} = A + B_D K_D$.

Since the blocks $K_{ii}$ of matrix $K_D$ have *arbitrary* nonzero patterns, it is necessary to develop a systematic procedure for computing this matrix. We begin by observing that a matrix $K_D$ with blocks of dimension $1 \times n_i$ can *always* be factorized as

$$K_D = \hat{K}_D C_D. \tag{29}$$

In this factorization, $\hat{K}_D = \text{diag}\{\hat{K}_{11}, \ldots, \hat{K}_{NN}\}$ represents a matrix composed of $N$ *full* diagonal blocks, and $C_D$ is formed from a subset of rows of the identity matrix.

The factorization in (29) allows us to treat the computation of $K_D$ as a decentralized static output feedback problem, in which $C_D$ plays the role of the output matrix. In view of that, we now proceed to describe an LMI-based procedure that can produce such a control law (the algorithm that follows represents a refinement of the method proposed in Zečević and Šiljak (2004)).

Since our immediate objective is to stabilize matrix $A$, we can temporarily disregard the nonlinear term $h(x)$. This adjustment leads to the following simplified LMI problem:

**Problem 2.** Minimize $a_1 \kappa_Y + a_2 \kappa_L$ subject to

$$Y_D > 0 \tag{30}$$

$$A Y_D + Y_D A^{\mathrm{T}} + B_D L_D + L_D^{\mathrm{T}} B_D < 0 \tag{31}$$

and

$$\begin{bmatrix} -\kappa_L I & L_D^{\mathrm{T}} \\ L_D & -I \end{bmatrix} < 0; \qquad \begin{bmatrix} Y_D & I \\ I & \kappa_Y I \end{bmatrix} > 0. \tag{32}$$

Since the gain matrix $K_D = L_D Y_D^{-1}$ must satisfy *arbitrary* structural constraints, matrices $L_D$ and $Y_D$ in Problem 2 should be computed in accordance with the following lemma.

**Lemma 3.** *Let us assume that Problem 2 is feasible with matrices $Y$ and $L$ of the form*

$$\begin{aligned} Y_D &= Y_1 + Y_2 \\ L_D &= L_0^D U_D^{\mathrm{T}} \end{aligned} \tag{33}$$

*where the components of $Y_D$ are given as*

$$\begin{aligned} Y_1 &= \rho Y_0^D + Q_D Y_Q^D Q_D^{\mathrm{T}} \\ Y_2 &= U_D Y_C^D U_D^{\mathrm{T}}. \end{aligned} \tag{34}$$

LMI *variables $L_0^D$, $Y_C^D$, and $Y_Q^D$ are assumed to be block-diagonal matrices, as are the constant terms $Y_0^D$, $Q_D$ and $U_D$. If conditions*

$$U_D = Y_0^D C_D^{\mathrm{T}} \tag{35}$$

*and*

$$Q_D^{\mathrm{T}} C_D^{\mathrm{T}} = 0 \tag{36}$$

*are satisfied, Problem 2 will produce a gain matrix of the form (29), and the closed-loop matrix $\tilde{A} = A + B_D \hat{K}_D C_D$ is guaranteed to be stable.*

**Proof.** The Sherman–Morrison lemma (*e.g.* Golub and van Loan (1996)) allows us to express $Y_D^{-1}$ as

$$Y_D^{-1} = (Y_1 + Y_2)^{-1} = (I - S_D R_D U_D^{\mathrm{T}}) Y_1^{-1}. \tag{37}$$

The matrices $S_D$ and $R_D$ in (37) are given as

$$S_D = Y_1^{-1} U_D Y_C^D \tag{38}$$

and

$$R_D = \left[ I + U_D^{\mathrm{T}} S_D \right]^{-1} \tag{39}$$

respectively. Since

$$Y_1^{-1} U_D = \rho^{-1} C_D^{\mathrm{T}} \tag{40}$$

by virtue of (35) and (36), it is easily verified that both of these matrices are block-diagonal.

Observing further that

$$L_D Y_D^{-1} = L_0^D (I - U_D^{\mathrm{T}} S_D R_D) U_D^{\mathrm{T}} Y_1^{-1} \tag{41}$$

and recalling (40), it follows that Problem 2 produces a block-diagonal gain matrix

$$K_D = L_D Y_D^{-1} = \hat{K}_D C_D \tag{42}$$

where

$$\hat{K}_D = \rho^{-1} L_0^D (I - U_D^{\mathrm{T}} S_D R_D). \tag{43}$$

Such a law clearly conforms to the factorization given in (29). □

It is important to recognize that in this case $Y_0^D$ *cannot* be chosen as the solution of a Lyapunov equation, since matrix $A$ is not block-diagonal in general. With that in mind, we propose to compute it by solving the following simple LMI problem in $\xi$ and $Y_0^D$ (in which $\beta$ is chosen so that $\hat{A} = A - \beta I$ is stable).

**Problem 3.** Minimize $\xi$ subject to

$$Y_0^D > 0 \tag{44}$$

$$\hat{A} Y_0^D + Y_0^D \hat{A}^{\mathrm{T}} < -I \tag{45}$$

and

$$\begin{bmatrix} -\xi I & Y_0^D \\ Y_0^D & -I \end{bmatrix} < 0. \tag{46}$$

Matrix $Y_0^D$ must be block-diagonal, with block sizes that are compatible with the block structure of $K_D$.

The motivation for the conditions set out in Problem 3 follows directly from inequality (31). Namely, if $Y_0^D$ is assumed to be a component of matrix $Y_D$, then the left-hand side of this inequality will contain the term $A Y_0^D + Y_0^D A^{\mathrm{T}}$, which satisfies

$$A Y_0^D + Y_0^D A^{\mathrm{T}} < -I + 2\beta Y_0^D \tag{47}$$

by virtue of condition (45). Inequality (46) ensures that $\left\| Y_0^D \right\|$ is minimized, which is conducive to the feasibility of the LMI optimization associated with Problem 2.

**Remark 3.** Note that inequality (31) involves only matrix $A$, and that $\hat{A} = A - \beta I$ plays an auxiliary role (it is used exclusively for the computation of an appropriate $Y_0^D$). As a result, when Problem 1 is solved using $\tilde{A} = A + B_D K_D$, it is *not* necessary to require $\alpha > \beta$. In that respect, Case 3 fundamentally differs from Case 1.

The following example illustrates the effectiveness of this design procedure.

**Example 1.** Let us consider a system of the form (1) where

$$A = \begin{bmatrix} -1.75 & 1.50 & 0 & 0 & 0 & 0 \\ 0.25 & -0.25 & 0.50 & 0.50 & -0.25 & 0.50 \\ 1.00 & 1.50 & -2.75 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.75 & -2.00 & 0 \\ 0.50 & 0.50 & -0.25 & -0.25 & -2.75 & -1.00 \\ 0 & 0 & 0 & 3.00 & -3.00 & -3.75 \end{bmatrix} \quad (48)$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}} \quad (49)$$

and $h(x)$ represents an uncertain nonlinearity that conforms to bound (2) with $H = I$. Matrix $A$ has a pair of unstable eigenvalues $\lambda_{1,2} = 0.268 \pm \mathrm{j}0.577$, and our objective in the following will be to design a robustly stabilizing control law which is subject to arbitrary information structure constraints. The specific nonzero pattern for the gain matrix was chosen to be

$$K = \begin{bmatrix} 0 & * & 0 & 0 & 0 & * \\ 0 & * & * & * & * & 0 \end{bmatrix}. \quad (50)$$

It should be noted that there is nothing special about this choice (in fact, we tested a number of other structures with similar results).

We begin by observing that $K$ can be partition in the manner described in (28), with

$$K_D = \left[ \begin{array}{ccc|ccc} 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 \end{array} \right] \quad (51)$$

and

$$K_C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & * \\ 0 & * & * & 0 & 0 & 0 \end{bmatrix}. \quad (52)$$

These components can now be computed separately, using the following procedures.

*The Design of $K_D$*

*Step* 1. Since $A$ is unstable, it is necessary to replace it with $\hat{A} = A - \beta I$ in (45). Setting $\beta = 1.5$, the corresponding $Y_0^D$ was found to be

$$Y_0^D = \begin{bmatrix} 0.31 & 0.10 & 0.07 & 0 & 0 & 0 \\ 0.10 & 0.48 & 0.13 & 0 & 0 & 0 \\ 0.07 & 0.13 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.47 & 0.05 & 0.16 \\ 0 & 0 & 0 & 0.05 & 0.27 & -0.16 \\ 0 & 0 & 0 & 0.16 & -0.16 & 0.37 \end{bmatrix}. \quad (53)$$

*Step* 2. Matrix $K_D$ can be factorized in the form (29), with

$$\hat{K}_D = \left[ \begin{array}{c|cc} * & 0 & 0 \\ 0 & * & * \end{array} \right] \quad (54)$$

and

$$C_D = \left[ \begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right]. \quad (55)$$

Given this factorization, the LMI procedure associated with Lemma 3 produces a stabilizing matrix

$$K_D = \left[ \begin{array}{ccc|ccc} 0 & -4.73 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4.62 & 3.65 & 0 \end{array} \right]. \quad (56)$$

The closed-loop eigenvalues in this case are: $\{-0.77 \pm \mathrm{j}0.48, -2.19 \pm \mathrm{j}0.68, -2.06, -4.03\}$, which allows us to use matrix $\tilde{A} = A + B_D K_D$ for computing $K_C$.

*The Design of $K_C$*

Matrix $K_C$ can be factorized in the manner indicated in (10), with

$$K_0 = \begin{bmatrix} 0 & 0 & * \\ * & * & 0 \end{bmatrix} \quad (57)$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (58)$$

Since $\tilde{A} = A + B_D K_D$ is stable, we can compute an appropriate matrix $Y_0$ by solving Lyapunov equation (24). Problem 1 (in which $A$ is replaced by $\tilde{A}$) then produces

$$K_C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.50 \\ 0 & -1.22 & -1.35 & 0 & 0 & 0 \end{bmatrix} \quad (59)$$

and $\alpha = 0.2791$.

**Case 4.** The design strategies associated with Cases 2 and 3 implicitly assume that matrix $A$ can be stabilized by some form of decentralized feedback. It is well known, however, that this assumption is not valid for systems with unstable modes that are *structurally fixed* with respect to decentralized control (see *e.g.* Šiljak (1991)). With that in mind, we now consider a class of control laws where this type of difficulty can be resolved.

We will begin by assuming that the gain matrix $K$ has at least one full column (*i.e.* a column that contains no zero elements). In the following, we will refer to states that correspond to such columns as "shared" states. Following (28) and (29), we can always decompose matrices of this type as

$$K = \hat{K}_D C_D + W \bar{C}_D + K_C \quad (60)$$

where $\bar{C}_D$ consists of the rows of $C_D$ that are associated with shared states, and $W$ is a full matrix of appropriate dimensions (as before, we will use $K_C$ to denote the remaining part of the gain matrix). Since $A$ is assumed to have one or more decentralized fixed modes, we now propose to precondition the optimization using a gain matrix of the form $\tilde{K} = \hat{K}_D C_D + W \bar{C}_D$. The first term in this expression represents the diagonal blocks of $K$, while the second one corresponds to its "full" columns. Such a law is obviously *not* decentralized, and therefore has the potential to stabilize matrix $A$. Sufficient conditions for the feasibility of this procedure are provided by the following lemma.

**Lemma 4.** *Suppose that Problem 2 is feasible with matrices $Y$ and $L$ of the form*

$$Y = Y_1 + Y_2$$
$$L_D = L_0^D U_D^{\mathrm{T}} \quad (61)$$

*where the components of $Y$ are given as*

$$Y_1 = \rho Y_0^D + Q_D Y_Q^D Q_D^{\mathrm{T}}$$
$$Y_2 = F_D Y_C F_D^{\mathrm{T}}. \quad (62)$$

*Matrices $Y_0^D$, $U_D$, $Q_D$ and $F_D$ are assumed to be constant and block-diagonal, while $\rho$, $L_0^D$, $Y_Q^D$, and $Y_C$ represent* LMI *variables. If conditions*

$$U_D = Y_0^D C_D^{\mathrm{T}} \quad (63)$$

$$F_D = Y_0^D \bar{C}_D^{\mathrm{T}} \quad (64)$$

*and*

$$Q_D^{\mathrm{T}} C_D^{\mathrm{T}} = 0 \quad (65)$$

*are satisfied, matrix A can be stabilized by a feedback law of the form*

$$\tilde{K} = \hat{K}_D C_D + W\bar{C}_D. \quad (66)$$

**Proof.** We begin by observing that in this case $Y_C$ is assumed to be a full matrix, which means that $Y$ will *not* be block-diagonal. Using condition (62) and the Sherman–Morrison lemma, we can express $Y^{-1}$ as

$$Y^{-1} = (Y_1 + Y_2)^{-1} = (I - SRF_D^{\mathrm{T}})Y_1^{-1} \quad (67)$$

where matrices $S$ and $R$ are given as

$$S = Y_1^{-1} F_D Y_C \quad (68)$$

and

$$R = \left[I + F_D^{\mathrm{T}} S\right]^{-1} \quad (69)$$

respectively (note that both of these matrices are full). The gain matrix produced by Problem 2 then becomes

$$LY^{-1} = L_0^D U_D^{\mathrm{T}} Y_1^{-1} - L_0^D U_D^{\mathrm{T}} SRF_D^{\mathrm{T}} Y_1^{-1}. \quad (70)$$

Observing that conditions (63)–(65) imply

$$U_D^{\mathrm{T}} Y_1^{-1} = \rho^{-1} C_D \quad (71)$$

and

$$F_D^{\mathrm{T}} Y_1^{-1} = \rho^{-1} \bar{C}_D^{\mathrm{T}} \quad (72)$$

we can directly rewrite (70) as

$$LY^{-1} = \hat{K}_D C_D + W\bar{C}_D \quad (73)$$

where

$$\hat{K}_D = \rho^{-1} L_0^D \quad (74)$$

and

$$W = -\rho^{-1} L_0^D U_D^{\mathrm{T}} SR. \quad (75)$$

The structure of $L_0^D$ ensures that $\hat{K}_D$ is a block-diagonal matrix, while the term $W\bar{C}_D$ corresponds to the "full" columns of $K$. □

Lemma 4 provides us with a systematic way for preconditioning the LMI optimization for a class of problems with decentralized fixed modes. Note that an appropriate choice for matrix $Y_0^D$ can be obtained by solving optimization Problem 3 (the procedure is the same as in Case 3). Once $\tilde{K}$ is computed, we can determine matrix $K_C$ by solving a variant of Problem 1 in which $A$ is replaced by $\tilde{A} = A + B_D\tilde{K}$.

## 4. Conclusions

In this paper a new strategy was proposed for formulating robust control laws with arbitrary information constraints. It was shown that the problem can be reduced to a variant of static output feedback design, which can be solved using linear matrix inequalities. In cases when the original system is unstable, it was found that some form of preconditioning is necessary in order for the optimization to be feasible. A number of generic scenarios were identified where such an approach is successful.

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