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# Cooperative Avoidance Control for Multiagent Systems

*The objective of this paper is to present a methodology for designing cooperative control laws for individual agents that guarantee collision avoidance in multiagent systems. The proposed avoidance control laws are easy to design and implement, and may be directly appended to the optimal control laws of the individual agents within the cooperation framework. The avoidance control laws are computed using value functions that resemble the behavior of barrier functions in the static optimization theory. The most attractive feature of the proposed optimization scheme is the fact that the avoidance laws are active only in the bounded sensing regions of each individual agent, and they do not interfere with the agents' individual optimal control laws outside of these regions.*
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## 1 Introduction

The concept of avoidance control has been originally formulated, and later studied in more detail, for the case of two independent agents in a noncooperative setting [1–6]. Avoidance control conditions were formulated to guarantee that the control actions of one of the agents will keep the system trajectories out of the prescribed unsafe set no matter what the control strategy of the other agent is. Thus, the avoidance conditions assumed the worst case scenario with no cooperation between the agents. Later, these results were formulated for the case of multiple agents that were separated into two groups [7]. One group consisted of agents that wanted to avoid the unsafe set, and the other group was comprised of agents whose goals were not known implying that the worst case scenario had to be assumed again. These control laws were designed for collision avoidance only, and the problem of combining them with the individual control laws of the agents' was not addressed. With the increasing interest in multiagent systems, the problem of establishing conditions for their safety verification has become important. In the case of the noncooperative scenario for a two-agent system, the problem of collision avoidance has been studied in Ref. [8], based on the ideas proposed in Ref. [9], using the level set methods [10,11] for computing solutions of Hamilton–Jacobi–Isaacs (HJI) partial differential equations (e.g., see Refs. [12,13] and references reported therein). Since these methods were related to the viscosity solutions of HJI equations, implying their high computational complexity, an efficient polytopic approximation method for computing guaranteed avoidance strategy for one of the two agents has been proposed in Refs. [14,15]. However, one of the open problems is how to efficiently apply these methods to multiagent scenarios even in the case of very simple dynamic models describing motions of the individual agents. In the context of multiagent control and coordination problem where the agents' motion is described by kinematic models, the collision avoidance was addressed using multiobjective and decentralized optimization methods in Ref. [16], heuristic methods in Ref. [17], stochastic optimization for choosing conflict-free optimal maneuvers in Ref. [18], navigation functions [19–21] in Ref. [22], and attractive/repulsive potentials in Refs. [23,24].

In this paper, we propose a method for control of multiagent systems, which is based on the avoidance control approach introduced in Ref. [1]. We shall describe the agents by using general nonlinear dynamic models and derive sufficient conditions for the collision-free maneuvers using functions that are active only in the bounded sensing regions containing the agents. Most importantly, this fact implies that the optimal control laws of individual agents remain unaffected outside of the sensing regions.

The organization of the paper is as follows. In Sec. 2, we describe the agents' behavior by a set of nonlinear differential equations and their individual goals to arrive at their desired targets by optimizing an infinite-horizon cost function. The avoidance value functions, which are active only in the bounded regions, are specified for each agent and added to the optimal value functions corresponding to their individual optimal control problems. Conditions that guarantee safe maneuvering of the agents are provided. The case when the agents are modeled using linear time-invariant models is considered in Sec. 3. The avoidance control laws are computed explicitly in terms of the solutions of the corresponding static matrix Riccati equations. In Sec. 4, we provide three illus-

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trative and challenging examples due to the assumption of symmetric initial and final conditions for the agents. In the first example, where the agents are represented by the kinematic model, the proof of the system safety validation for an arbitrary number of agents is provided. In addition, we show that the appropriate choice of safety and avoidance regions breaks the symmetry of the problem. In the second example, we constrain the agents' feedback control laws to be norm bounded, which is desirable for practical applications. Finally, the third example is chosen in order to illustrate the application of the methodology when the state variables include velocities of the agents.

## 2 Cooperative Avoidance Control

Let us assume a group of $N$ independent agents whose individual motions are described by the following dynamic models:

$$\dot{x}_i = f_i(x_i, u_i) \quad \forall\, i \in \mathbf{N} = \{1, 2, \ldots, N\} \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$ is the state, and $u_i \in \mathbb{R}^{m_i}$ is the control input of the $i$th agent. The $n_i$-dimensional vector functions $f_i(\cdot,\cdot)$, $i \in \mathbf{N}$, are assumed to be continuously differentiable with respect to both arguments. The above equation is assumed to be valid at every time instance $t \in \mathbf{T} = [0, +\infty)$. The agents' inputs are assumed to belong to the set of feedback strategies, that is, $u_i \in \mathcal{U}_i = \{\phi_i(\cdot) : \mathbb{R}^n \to \mathbb{R}^{m_i}\}$. Notice that $u_i(\cdot)$ is assumed to be a function of the overall state space vector $x = [x_1^T, \ldots, x_N^T]^T \in \mathbb{R}^n$ to accommodate the later analysis that would treat collision avoidance behavior of the agents. With each agent, we associate a cost function of the following form:

$$J_i = \int_0^\infty q_i(x_i, u_i)\,dt \quad \forall\, i \in \mathbf{N} \tag{2}$$

We assume that the above cost functions define the strategies of agents to arrive at their target points denoted as $x_i^e \in \mathbb{R}^{n_i}$, for all $i \in \mathbf{N}$. In order for this problem to be well defined, it is necessary to assume that the target points are in the reachable sets of the corresponding agents. Additionally, we assume that the optimal value functions $v_i^o(x_i)$, $i \in \mathbf{N}$, along the optimal trajectories, satisfy the following equation:

$$v_i^o[x_i(t)] = \int_t^\infty q_i\{x_i(\tau), u_i^o[x_i(\tau)]\}\,d\tau \quad \forall\, t \in \mathbf{T} \quad \forall\, i \in \mathbf{N} \tag{3}$$

and belong to the class $C^1$ of continuously differentiable functions. The agents' control strategies denoted by $u_i^o(x_i)$, for all $i \in \mathbf{N}$, are optimal control laws that are obtained by the following minimizations:

$$u_i^o(x_i) = \arg\min_{u_i \in \mathcal{U}_i} \left\{ q_i(x_i, u_i) + \frac{\partial v_i^o}{\partial x_i} f_i(x_i, u_i) \right\} \quad \forall\, i \in \mathbf{N} \tag{4}$$

The agents' Hamiltonian functions are defined for the general functions $u_i(\cdot)$ and $v_i(\cdot)$, in a standard way [25], as follows:

$$H_{f_i}\left(x_i, \frac{\partial v_i}{\partial x_i}, u_i\right) = q_i(x_i, u_i) + \frac{\partial v_i}{\partial x_i} f_i(x_i, u_i) \quad \forall\, i \in \mathbf{N} \tag{5}$$

The optimal value functions and the optimal feedback control laws are assumed to satisfy the corresponding Hamilton–Jacobi equations:

$$q_i[x_i, u_i^o(x_i)] + \frac{\partial v_i^o}{\partial x_i} f_i[x_i, u_i^o(x_i)] = 0 \quad \forall\, i \in \mathbf{N} \tag{6}$$

The cost functions are non-negative functions, that is, $q_i(x_i, u_i) \geqslant 0$ such that $q_i[x_i, u_i^o(x_i)] > 0$, for all $x_i \neq x_i^e$ that are in the reachable set of the $i$th agent. The target points satisfy the equilibrium equations:

$$f_i[x_i^e, u_i^o(x_i^e)] = 0 \quad \forall\, i \in \mathbf{N} \tag{7}$$

From Eq. (6), it follows that

$$\frac{dv_i^o}{dt} = \frac{\partial v_i^o}{\partial x_i} f_i[x_i, u_i^o(x_i)] = -q_i[x_i, u_i^o(x_i)] < 0, \ \forall\, x_i \neq x_i^e, \ \forall\, i \in \mathbf{N} \tag{8}$$

where $\partial v_i^o / \partial x_i$ is a row vector of dimension $n_i$, which denotes the gradient of $v_i^o$ with respect to $x_i$. Thus, from Eq. (8), it follows that the value functions may be used as Liapunov functions in order to establish convergence of the agents' trajectories to their corresponding equilibria.

Now, our goal is to be able to guarantee that the agents would arrive at their target/equilibrium points without collisions. In order to do so, for each pair of agents, we define the following functions:

$$v_{ij}(x_i, x_j) = \left( \min\left\{ 0, \frac{\|x_i - x_j\|^2 - R^2}{\|x_i - x_j\|^2 - r^2} \right\} \right)^2 \quad i, j \in \mathbf{N} \quad i \neq j \tag{9}$$

where $R > r > 0$. The symbol $R$ denotes the radius of the region in which agents can detect the presence of the other agents. The lower case $r$ denotes the avoidance region, that is, the smallest safe distance between the vehicles. Notice that the functions defined in Eq. (9) may be linked to the penalty and barrier functions used in the static optimization literature [26]. The partial derivative of $v_{ij}(x_i, x_j)$ with respect to $x_i$ is given by

$$\frac{\partial v_{ij}}{\partial x_i} = \begin{cases} 0 & \text{if } \|x_i - x_j\| \geqslant R \\ 4\dfrac{(R^2 - r^2)(\|x_i - x_j\|^2 - R^2)}{(\|x_i - x_j\|^2 - r^2)^3}(x_i - x_j)^T & \text{if } R > \|x_i - x_j\| > r \\ \text{not defined} & \text{if } \|x_i - x_j\| = r \\ 0 & \text{if } \|x_i - x_j\| < r \end{cases} \tag{10}$$

Since the functions $v_{ij}(x_i, x_j)$ are symmetric with respect to their arguments, the partial derivative with respect to $x_j$ may be obtained directly from Eq. (10) by swapping the indices $i$ and $j$. Also, in order to make our presentation simpler, we have chosen that full state vectors $x_i$, $i \in \mathbf{N}$, represent agents' positions. In general, only subsets of the state variables, as elements of the agents' state vectors, represent the positions of the agents but the analysis would stay the same [7]. For more information on partial stability and stabilization results that would guarantee straightforward extension of our results for this general case, we refer to a recent survey paper [27] and the references reported therein. In addition, an example is provided in Sec. 4.3 that illustrates the application of our methodology when agents' velocities are treated as state variables.

In order to achieve both goals at the same time, that is, to avoid collisions while the agents converge to their target points, we propose the following functions:

$$v_i(x) = \sum_{j=1}^{N} v_{ij}(x_i, x_j) \quad \forall\, i \in \mathbf{N} \tag{11}$$

where $v_{ii}(x_i, x_i) = v_i^o(x_i)$. Notice that the above value functions may be treated as components of the overall vector Liapunov-type function [28–33]. The agents' avoidance control laws are obtained by minimizing the Hamiltonians with respect to the value functions (11), that is,

$$u_i^a(x) = \arg\min_{u_i \in \mathcal{U}_i} \left\{ q_i(x_i, u_i) + \frac{\partial v_i}{\partial x_i} f_i(x_i, u_i) \right\} \quad \forall\, i \in \mathbf{N} \tag{12}$$

One important feature of the avoidance control laws is that outside of the detection regions, these control laws coincide with the optimal control laws of the independent agents, that is,

$$u_i^a(x) = u_i^a(x_i) \quad \text{if } \|x_i - x_j\| \geq R, \ \forall j \in \mathbf{N} \quad j \neq i \tag{13}$$

Since the avoidance components of the control laws are active only in the detection region, they may be considered as local, which is compatible with the nature of the collision avoidance problem.

In order to be able to prove collision avoidance for the group of $N$ agents, we combine the individual agent dynamics into the overall dynamic system

$$\dot{x} = f(x,u) \tag{14}$$

where $x=[x_1^T,x_2^T,\ldots,x_N^T]^T \in \mathbb{R}^n$ is the state, and $u=[u_1^T,u_2^T,\ldots,u_N^T]^T \in \mathbb{R}^m$ is the input. The agents' state and input vectors are assumed to be disjoint, that is, $n=\sum_{i=1}^N n_i$ and $m=\sum_{i=1}^N m_i$. The assumption that $f_i(\cdot,\cdot) \in C^1$, for all $i \in \mathbf{N}$, implies that the concatenated $n$-dimensional function is also a continuously differentiable function with respect to both arguments. With the overall system, we associate the cost function that is the sum of the individual costs, that is,

$$J = \sum_{i=1}^N J_i = \int_0^\infty q(x,u)dt, \ q(x,u) = \sum_{i=1}^N q_i(x_i,u_i) \tag{15}$$

For the overall system, we define the avoidance region by defining the avoidance sets for each pair of agents as

$$\Omega_{ij} = \{x : x \in \mathbb{R}^n, \|x_i - x_j\| \leq r\} \tag{16}$$

and the overall system sensing or detection region by defining pairwise detection regions

$$\mathcal{D}_{ij} = \{x : x \in \mathbb{R}^n, \|x_i - x_j\| \leq R\} \tag{17}$$

Then, the overall avoidance region is given by

$$\Omega = \bigcup_{i,j \in \mathbf{N}, j>i} \Omega_{ij} \tag{18}$$

and the overall detection region is defined by

$$\mathcal{D} = \bigcup_{i,j \in \mathbf{N}, j>i} \mathcal{D}_{ij} \tag{19}$$

At this point, we recall the following definition for the avoidance of the set $\Omega$ [1,3].

DEFINITION 1. *The dynamic system $\dot{x}=f[x,u(x)]$ avoids $\Omega$, $\Omega \subset \mathbb{R}^n$, if and only if for each solution $x(t,x_0)$, $t \in \mathbf{T}=[0,+\infty)$, $x_0 \notin \Omega$ implies $x(t,x_0) \notin \Omega$ for all $t \in \mathbf{T}$.*

In order to be able to establish conditions for the avoidance of the set $\Omega$, we follow the methodology proposed in Refs. [1,3–6]. Similarly, as in the case of avoidance sets, we define the safety region for each pair of agents as

$$\Gamma_{ij} = \{x : x \notin \Omega, r < \|x_i - x_j\| \leq \bar{r}\} \tag{20}$$

where $R \geq \bar{r} > r$. It is important to note that the above definition of safety regions is a special case of the general definition used in Refs. [1,3–7] where the upper bound on the $\|x_i-x_j\|$ does not have to be a constant. This has been done for the simplicity of presentation purposes yet it is easy to show that the definitions lead to the equivalent result in terms of the existence of appropriate safety regions that would guarantee collision-free coordination of the agents. The overall safety region is defined as the union of the pairwise safety regions, that is,

$$\Gamma = \bigcup_{i,j \in \mathbf{N}, j>i} \Gamma_{ij} \tag{21}$$

As the composite value or Liapunov-type function, we choose the following:

$$v(x) = \sum_{i=1}^N \sum_{j=i}^N v_{ij}(x_i,x_j) \tag{22}$$

Now, the overall Hamiltonian is defined as [25]:

$$H_f\left(x, \frac{\partial v}{\partial x}, u\right) = q(x,u) + \frac{\partial v}{\partial x} f(x,u) \tag{23}$$

In the next theorem, we formulate a cooperative avoidance result for the collision-free coordination of the agents. The cooperation is validated by proving that the agents can cooperatively decrease the composite Liapunov-type function in Eq. (22) by decreasing its individual value functions given in Eq. (11).

THEOREM 1. *If the following inequality is satisfied:*

$$\sum_{i=1}^N H_{f_i}\left[x_i, \frac{\partial v_i}{\partial x_i}, u_i^a(x)\right] - \sum_{i=1}^N q_i[x_i, u_i^a(x)] \leq 0 \quad \forall x \in \Gamma \tag{24}$$

*then the set $\Omega$ is avoidable for the system (14) with the subsystem control strategies $u_i^a(x)$ defined in Eq. (12).*

*Proof.* We start by computing the time derivative of the composite function in the region $\Gamma$ as follows:

$$\begin{aligned}\frac{dv}{dt} &= \sum_{i=1}^N \sum_{j=i}^N \frac{dv_{ij}}{dt} = \sum_{i=1}^N \frac{dv_{ii}}{dt} + \sum_{i=1}^N \sum_{j>i} \frac{dv_{ij}}{dt} \\ &= \sum_{i=1}^N \frac{dv_{ii}}{dt} + \sum_{i=1}^N \sum_{j>i} \left\{ \frac{\partial v_{ij}}{\partial x_i} f_i[x_i,u_i^a(x)] + \frac{\partial v_{ij}}{\partial x_j} f_j[x_j,u_j^a(x)] \right\} \\ &= \sum_{i=1}^N \frac{dv_{ii}}{dt} + \sum_{i=1}^N \sum_{j>i} \frac{\partial v_{ij}}{\partial x_i} f_i[x_i,u_i^a(x)] + \sum_{i=1}^N \sum_{j>i} \frac{\partial v_{ij}}{\partial x_j} f_j[x_j,u_j^a(x)]\end{aligned} \tag{25}$$

Now, notice that

$$\begin{aligned}\sum_{i=1}^N \sum_{j>i} \frac{\partial v_{ij}}{\partial x_j} f_j[x_j,u_j^a(x)] &= \sum_{j=1}^N \sum_{i<j} \frac{\partial v_{ij}}{\partial x_j} f_j[x_j,u_j^a(x)] \\ &= \sum_{i=1}^N \sum_{j<i} \frac{\partial v_{ji}}{\partial x_i} f_i[x_i,u_i^a(x)] \\ &= \sum_{i=1}^N \sum_{j<i} \frac{\partial v_{ij}}{\partial x_i} f_i[x_i,u_i^a(x)]\end{aligned} \tag{26}$$

Since $v_{ij}(x_i,x_j)=v_{ji}(x_j,x_i)$, we finally obtain

$$\begin{aligned}\frac{dv}{dt} &= \sum_{i=1}^N \frac{dv_{ii}}{dt} + \sum_{i=1}^N \sum_{j>i} \frac{\partial v_{ij}}{\partial x_i} f_i[x_i,u_i^a(x)] + \sum_{i=1}^N \sum_{j<i} \frac{\partial v_{ij}}{\partial x_i} f_i[x_i,u_i^a(x)] \\ &= \sum_{i=1}^N \sum_{j=1}^N \frac{\partial v_{ij}}{\partial x_i} f_i[x_i,u_i^a(x)] \\ &= \sum_{i=1}^N \frac{\partial v_i}{\partial x_i} f_i[x_i,u_i^a(x)] = \sum_{i=1}^N H_{f_i}\left[x_i, \frac{\partial v_i}{\partial x_i}, u_i^a(x)\right] - \sum_{i=1}^N q_i[x_i,u_i^a(x)] \\ &\leq 0 \quad \forall x \in \Gamma\end{aligned} \tag{27}$$

Thus, we proved that the function $v(x)$ is nonincreasing in the region $\Gamma$. Also, notice that the values of the function $v(x)$ are finite for the finite values of its argument $x$ that are outside of the region $\Omega$, that is, when $x \in \Omega^c$, where $\Omega^c=\mathbb{R}^n \setminus \Omega$ denotes the set complement of $\Omega$. Due to the continuity of solutions of the system (14), the assumption that the initial condition satisfies $x_0 \in \Omega^c$, and that the following conditions hold:

$$\lim_{\|x_i-x_j\| \to r+} v_{ij}(x_i,x_j) = +\infty \quad \forall i,j \in \mathbf{N} \quad i \neq j \tag{28}$$

we conclude that $x(t,x_0)$ will never enter $\Omega$. In the above equation, $\|x_i-x_j\| \to r+$ denotes convergence to $r$ from above, that is, $\|x_i-x_j\|=r+\delta$ while $\delta \to 0$ and $\delta \geq 0$. Q.E.D.

In order to guarantee that the agents would arrive at the corresponding equilibria, a stronger set of conditions is needed. These conditions are formulated in the following theorem.

THEOREM 2. *If* $x^e=[(x_1^e)^T,(x_2^e)^T,\ldots,(x_N^e)^T]^T \notin \mathcal{D}$ *and* $H_f[x,\partial v/\partial x,u^a(x)]-q[x,u^a(x)]<0$ *for all* $x\in\Omega^c$, $x\neq x^e$ *then the set* $\Omega$ *is avoidable for the system* (14), *and all the agents will arrive to their target/equilibrium points* $x_i^e$ *with the subsystem control strategies defined in Eq.* (12).

*Proof.* First, notice that the condition $x^e=[(x_1^e)^T,(x_2^e)^T,\ldots,(x_N^e)^T]^T\notin\mathcal{D}$ implies that the non-negative Liapunov-type function $v(x)$, defined in Eq. (22), equals zero only at the equilibrium point $x^e$. Then, the condition $dv(x)/dt=H_f[x,\partial v/\partial x,u^a(x)]-q[x,u^a(x)]<0$ guarantees convergence to the equilibrium by the standard Liapunov analysis (e.g., see Ref. [34]) and Theorem 1 guarantees collision-free coordination since the system trajectories cannot enter the avoidance region $\Omega$. Q.E.D.

It is important to note that the avoidance functions may be defined to be much more general than those in Eq. (9). As an illustration, let us consider the following functions:

$$v_{ij}(x_i,x_j)=\left(\min\left\{0,\frac{\beta_{ij}(x_i,x_j)-b_{ij}}{\alpha_{ij}(x_i,x_j)-a_{ij}}\right\}\right)^2 \quad i,j\in\mathbf{N}\quad i\neq j \tag{29}$$

where $a_{ij}$ and $b_{ij}$, $i,j\in\{1,2,\ldots,N\}$, $i\neq j$, are positive numbers. Functions $\alpha_{ij}(x_i,x_j)$ and $\beta_{ij}(x_i,x_j)$, $i,j\in\{1,2,\ldots,N\}$, $i\neq j$, are chosen to be continuously differentiable and non-negative functions of their arguments such that $x_i=x_j$ implies $\alpha_{ij}(x_i,x_i)=\beta_{ij}(x_i,x_i)=0$ and

$$\{x_i,x_j|\alpha_{ij}(x_i,x_j)\leq a_{ij}\}\subset\{x_i,x_j|\beta_{ij}(x_i,x_j)<b_{ij}\} \tag{30}$$

with inclusion being strict. Now, one may easily show that the more general detection region for the $i$th agent with respect to the $j$th agent may be defined (in terms of the agents' states only) as the following nonempty set:

$$\mathcal{D}_{ij}=\{x_i,x_j|\beta_{ij}(x_i,x_j)\leq b_{ij}\} \tag{31}$$

and the corresponding avoidance region as

$$\Omega_{ij}=\{x_i,x_j|\alpha_{ij}(x_i,x_j)\leq a_{ij}\} \tag{32}$$

Similar to Eq. (10), one may compute the partial derivative $\partial v_{ij}/\partial x_i$ in the sensing region $\mathcal{D}_{ij}$ as

$$\frac{\partial v_{ij}}{\partial x_i}=2\frac{\beta_{ij}-b_{ij}}{(\alpha_{ij}-a_{ij})^3}\left[(\alpha_{ij}-a_{ij})\frac{\partial\beta_{ij}}{\partial x_i}-(\beta_{ij}-b_{ij})\frac{\partial\alpha_{ij}}{\partial x_i}\right] \tag{33}$$

and note that the function $v_{ij}(\cdot,\cdot)$ is continuously differentiable outside of the avoidance region $\Omega_{ij}$ and zero outside of the sensing region $\mathcal{D}_{ij}$. Also, these general functions need not to be symmetric with respect to their arguments, that is, $v_{ij}(x_i,x_j)\neq v_{ji}(x_i,x_j)$. In this paper, we will assume the symmetry condition holds if not stated otherwise. The usage of an additional degree of freedom by choosing nonsymmetric avoidance functions will be illustrated in the section dedicated to examples.

## 3 Linear Systems With the Quadratic Cost

In order to show how the avoidance control laws in Eq. (12) can be computed explicitly, we assume that the behavior of each agent can be modeled as a linear time-invariant dynamic system in the following form:

$$\dot{x}_i=A_ix_i+B_iu_i\quad\forall i\in\mathbf{N} \tag{34}$$

Again, we assume that each agent's goal is to arrive at the equilibrium point $x_i^e\in\mathbb{R}^{n_i}$ that satisfies the equilibrium equation (linear equivalent of Eq. (7))

$$A_ix_i^e+B_iu_i^e=0 \tag{35}$$

With each equilibrium, we associate the quadratic cost

$$q_i(x_i,u_i)=\frac{1}{2}[(x_i-x_i^e)^TQ_i(x_i-x_i^e)+(u_i-u_i^e)^TR_i(u_i-u_i^e)] \tag{36}$$

where $R_i$ is a positive definite symmetric matrix and $Q_i$ is a non-negative definite symmetric matrix, for all $i\in\mathbf{N}$. In addition, the pair $(A_i,B_i)$ is assumed to be stabilizable and the pair $(A_i,Q_i^{1/2})$ is assumed to be detectable. Then, this infinite-horizon problem has unique stabilizing solution in terms of the solution $P_i$ of the algebraic matrix Riccati equation (e.g., see Ref. [35])

$$P_iA_i+A_i^TP_i-P_iB_iR_i^{-1}B_i^TP_i+Q_i=0\quad\forall i\in\mathbf{N} \tag{37}$$

The optimal cost to go function is known to be equal to

$$v_i^o(x_i)=\frac{1}{2}(x_i-x_i^e)^TP_i(x_i-x_i^e) \tag{38}$$

and the optimal input is computed using the equation

$$\begin{aligned} u_i^o(x_i)&=\arg\min_{u_i\in\mathcal{U}_i}\left\{\frac{1}{2}[(x_i-x_i^e)^TQ_i(x_i-x_i^e)+(u_i-u_i^e)^TR_i(u_i-u_i^e)]+\frac{\partial v_i^o}{\partial x_i}(A_ix_i+B_iu_i)\right\}\\ &=\arg\min_{u_i\in\mathcal{U}_i}\left\{\frac{1}{2}(u_i-u_i^e)^TR_i(u_i-u_i^e)+(x_i-x_i^e)^TP_iB_i(u_i-u_i^e)+(x_i-x_i^e)^TP_iA_i(x_i-x_i^e)+\frac{1}{2}(x_i-x_i^e)^TQ_i(x_i-x_i^e)\right\} \end{aligned} \tag{39}$$

as

$$u_i^o(x_i)=-R_i^{-1}B_i^TP_i(x_i-x_i^e)+u_i^e \tag{40}$$

Now,

$$\begin{aligned} u_i^a(x)&=\arg\min_{u_i\in\mathcal{U}_i}\left\{\frac{1}{2}[(x_i-x_i^e)^TQ_i(x_i-x_i^e)+(u_i-u_i^e)^TR_i(u_i-u_i^e)]+\frac{\partial v_i}{\partial x_i}(A_ix_i+B_iu_i)\right\}\\ &=\arg\min_{u_i\in\mathcal{U}_i}\left\{\frac{1}{2}(u_i-u_i^e)^TR_i(u_i-u_i^e)+\frac{\partial v_i}{\partial x_i}B_i(u_i-u_i^e)+\frac{\partial v_i}{\partial x_i}A_i(x_i-x_i^e)+\frac{1}{2}(x_i-x_i^e)^TQ_i(x_i-x_i^e)\right\} \end{aligned} \tag{41}$$

producing

$$\begin{aligned} u_i^a(x)&=u_i^e-R_i^{-1}B_i^T\frac{\partial v_i^T}{\partial x_i}=u_i^e-R_i^{-1}B_i^T\sum_{j=1}^{N}\frac{\partial v_{ij}^T}{\partial x_i}\\ &=u_i^e-R_i^{-1}B_i^TP_i(x_i-x_i^e)-R_i^{-1}B_i^T\sum_{j\neq i}\frac{\partial v_{ij}^T}{\partial x_i}\\ &=u_i^o(x_i)-R_i^{-1}B_i^T\sum_{j\neq i}\frac{\partial v_{ij}^T}{\partial x_i} \end{aligned} \tag{42}$$

Again, from the above equation the relation between $u_i^a(x)$ and $u_i^o(x_i)$, given in Eq. (13), is even more straightforward. In other words, the components related to the cooperative avoidance of collisions are active only in the local detection regions of the agents. Outside of these regions, the motions of the agents are governed by the optimal control laws that will lead them to their target/equilibrium points.

## 4 Examples

In this section, we present three illustrative examples. In the first one, we show how to use the flexibility of choosing $\alpha_{ij}(\cdot,\cdot)$ and $\beta_{ij}(\cdot,\cdot)$ functions in Eq. (29) in order to solve problems with symmetrical initial and final conditions. These problems are especially difficult to solve using gradient based methods. One of the ways to solve these problems is to break the symmetry using different shapes for the avoidance regions and choose appropriate functions having level sets that define the boundaries of the avoidance regions.

In the second example, the feedback strategies of the agents are assumed to be norm bounded. The agents' dynamic models are linear. The norm bounds, the avoidance, and the safety regions are not the same for all agents. Thus, this example represents the scenario with *heterogeneous* agents.

In the third example, we consider agents with second order dynamics in which the state contains both position and velocity. This illustrates that our approach is not limited to cases where the state is just comprised of the positions of the agents.

It is important to note that the methodology proposed in this paper is *not restrictive* to any particular design of feedback control laws. As long as the time derivative of the overall value function (22) is nonpositive, any corresponding controller design will guarantee collision-free scenario. Thus, the choice of the optimal control framework is mainly for presentation purposes.

**4.1 Example 1: Kinematic Model With Unbounded Inputs.** As an illustration of the proposed methodology for designing the cooperative avoidance control laws that would guarantee the collision-free motion of the agents, we assume in this first example that the agents are represented by the basic kinematic model

$$\dot{x}_i = u_i \quad \forall\, i \in \mathbf{N} \tag{43}$$

and that each agent is targeting its own equilibrium $x_i^e$. Since $A_i=0$ and $B_i=I$, from Eq. (35), it follows that $u_i^e=0$. If we choose the quadratic cost such that $Q_i=R_i=I$, then the Riccati equation is given by

$$-P_i^2 + I = 0 \tag{44}$$

We choose the positive definite solution $P_i=I$, which implies that

$$v_i^o(x_i) = v_{ii}(x_i,x_i) = \frac{1}{2}\|x_i - x_i^e\|^2 \quad \forall\, i \in \mathbf{N} \tag{45}$$

and compute the optimal control laws as

$$u_i^o(x_i) = -(x_i - x_i^e) + u_i^e = -x_i + x_i^e = -\frac{\partial v_{ii}^T}{\partial x_i} \quad \forall\, i \in \mathbf{N} \tag{46}$$

Using the fact that $R_i=B_i=I$, we obtain using Eq. (42)

$$u_i^a(x) = u_i^o(x_i) - \sum_{j\neq i} \frac{\partial v_{ij}^T}{\partial x_i} = -\sum_{j=1}^{N} \frac{\partial v_{ij}^T}{\partial x_i} \quad \forall\, i \in \mathbf{N} \tag{47}$$

Finally, using Eqs. (27), (43), and (47), we obtain the following:

$$\frac{dv}{dt} = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial v_{ij}}{\partial x_i} f_i[x_i,u_i^a(x)] = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial v_{ij}}{\partial x_i} u_i^a(x)$$
$$= \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\frac{\partial v_{ij}}{\partial x_i}\right)\left(-\sum_{j=1}^{N}\frac{\partial v_{ij}}{\partial x_i}\right)^T = -\sum_{i=1}^{N}\left\|\sum_{j=1}^{N}\frac{\partial v_{ij}}{\partial x_i}\right\|^2 \leq 0 \tag{48}$$

Now, since

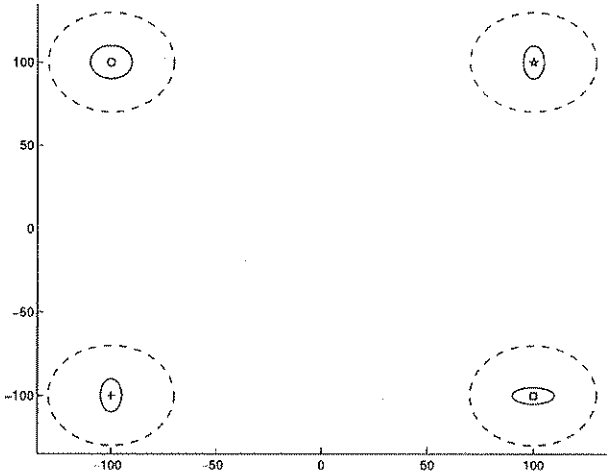


**Fig. 1 Initial configuration**

$$\frac{dv}{dt} = H_f\left[x,\frac{\partial v}{\partial x},u^a(x)\right] - q[x,u^a(x)]$$
$$= \sum_{i=1}^{N} H_{f_i}\left[x_i,\frac{\partial v_i}{\partial x_i},u_i^a(x)\right] - \sum_{i=1}^{N} q_i[x_i,u_i^a(x)] \tag{49}$$

the conditions of Theorem 1 are satisfied for any properly chosen avoidance and safety regions, that is, $\Omega$ and $\Gamma$, respectively. Also, notice that Eq. (48) implies that $\Gamma = \mathcal{D}\backslash\Omega$.

As an example of our results, let us consider four agents located at the vertices of a square with coordinates (100,100), (100, −100), (−100,−100), and (−100,100), as depicted in Fig. 1. The destination for each agent is located at the opposite vertex of the square. For this problem, we chose the safety regions to be circular with a uniform radius $R=\bar{r}=30$ (denoted by dashed circles). Notice that the equality $R=\bar{r}$ is implied by Eq. (48). However, we chose the detection and the avoidance regions to be ellipsoidal for the first three agents, with various orientation, and circular for the fourth agent (all denoted by full-lined circles). In order to do this, we can define the functions $\alpha_{ij}(\cdot,\cdot)$ in Eq. (29) as follows:

$$\alpha_{ij}(x_i,x_j) = (x_i - x_j)^T M_i (x_i - x_j) \quad i,j \in \{1,\ldots,4\} \quad i \neq j$$

where $M_1=M_3=\left[\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right]$, $M_2=\left[\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right]$, and $M_4=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$. We also chose the constants $a_{ij}=20^2$, $\forall i,j=1,\ldots,4$, which will guarantee that the agents will not come closer than a distance of 20 units to each other.

In Fig. 2, the agents are moving in straight lines toward their corresponding equilibria, according to the optimal control inputs $u_i^o(x_i)$ given in Eq. (46), until they detect each other on the boundary of the safety regions. This is when the avoidance control inputs get activated in Eq. (47) and the agents begin to resolve the conflict as seen in Figs. 3 and 4. Once the conflict has been resolved the agents start heading toward their corresponding equilibrium points in an optimal fashion as depicted in Fig. 5. The complete trajectories of the agents are shown in Fig. 6 with final positions placed at (−100,−100), (−100,100), (100,100), and (100,−100), respectively. Notice that in all figures the safety and avoidance regions are plotted only for the last time sample. In addition to the collision-free condition (48), the derivative of the value function was never zero, which in our case means it was always negative except at the equilibrium. Thus, the convergence to the agents' equilibria is to be expected due to Theorem 2.

**4.2 Example 2: Linear Dynamics With Input Constraints.** In this example, we assume that the agents are governed by the following linear model:

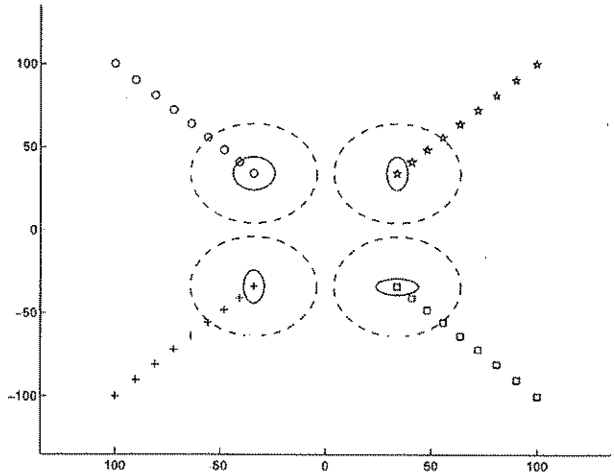


Fig. 2 Agents start moving toward the equilibrium

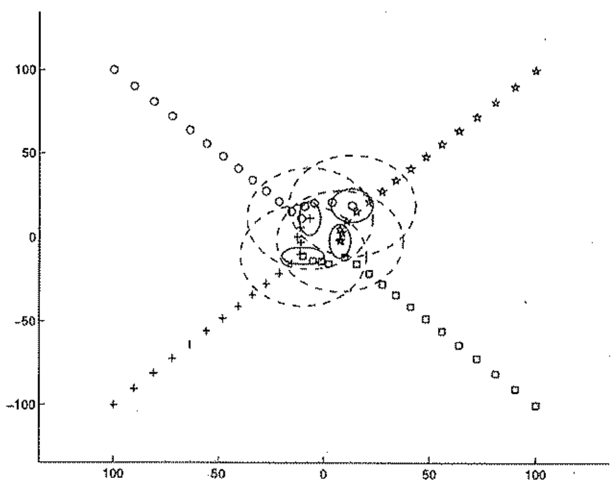


Fig. 3 Agents starting to resolve the conflict

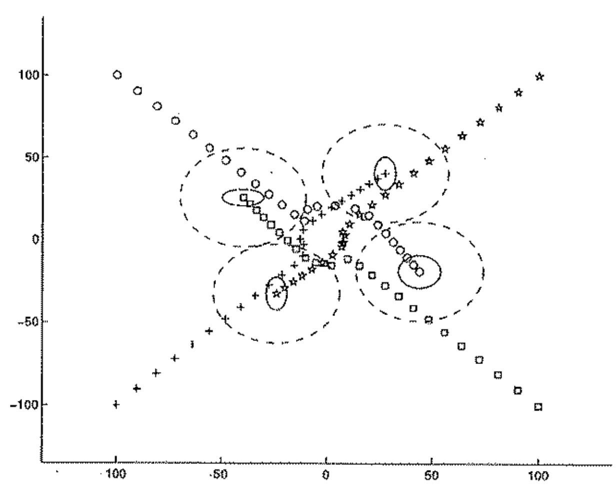


Fig. 4 Conflict resolved

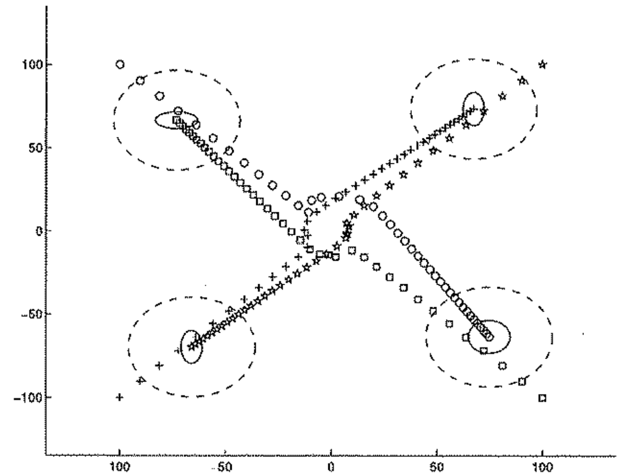


Fig. 5 Agents continuing toward the equilibrium

$$\dot{x}_i = -x_i + u_i \quad \forall i \in \{1,2,3,4\} \tag{50}$$

where $A_i = -I$ and $B_i = I$ in Eq. (34).

The avoidance and detection regions for agents 1, 2, and 4 are defined exactly as in Sec. 4.1. However, for agent 3, we redefined the detection and avoidance regions to be ellipsoidal, that is,

$$\alpha_{3j}(x_3, x_j) = \beta_{3j}(x_3, x_j) = (x_3 - x_j)^T M_3 (x_3 - x_j)$$

where $M_3 = \left[\begin{smallmatrix} 4 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ for $j \in \{1,2,4\}$. The constants $a_{3j}$ and $b_{3j}$ are chosen to be $20^2$ and $30^2$, respectively.

It is easy to verify that the control law

$$u_i^o(x_i) = -(\sqrt{2} - 1)(x_i - x_i^e) + u_i^e \tag{51}$$

is optimal and satisfies the algebraic Riccati equation (37) with $P_i = (\sqrt{2} - 1)I$, where in this case we also chose $Q_i = R_i = I$. The input at the equilibrium can be also computed using Eq. (35) as $u_i^e = x_i^e$.

Then, using Eqs. (51) and (29), we design the avoidance control laws $u_i^a$ as given in Eq. (42). We further assume that the norms of the input vectors $u_i^a$ are bounded by constants $m_i$ for $i = 1, \ldots, 4$ that we chose to be 250, 200, 190, and 280, respectively, that is

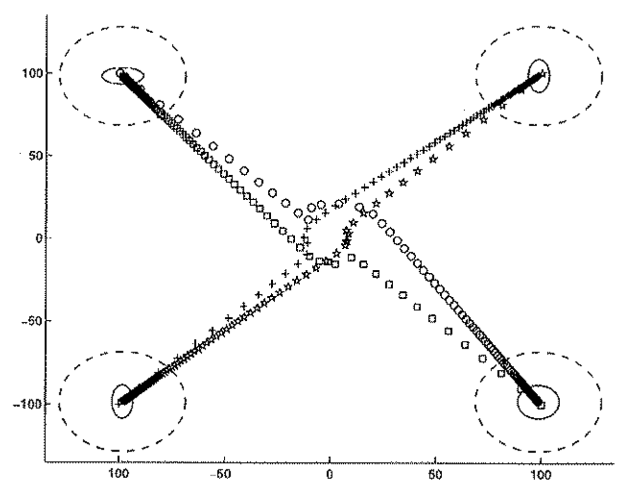


Fig. 6 Initial to final configuration

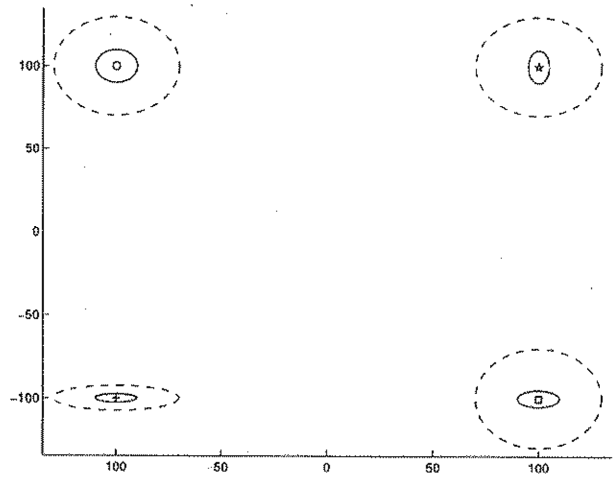


**Fig. 7 Initial configuration**

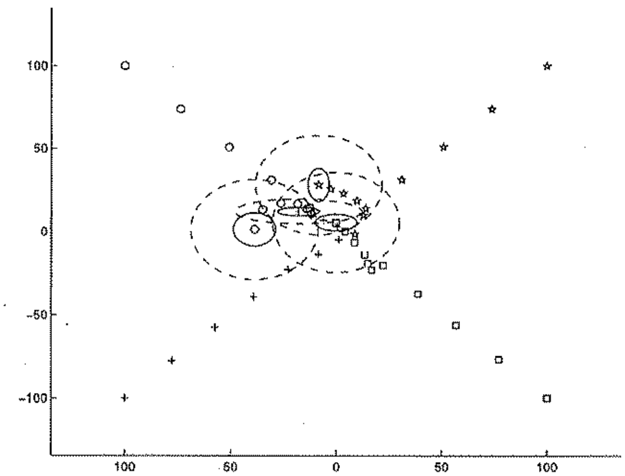


**Fig. 9 Agents starting to resolve the conflict**

$$u_i^t(x) = \begin{cases} m_i \dfrac{u_i^a(x)}{\|u_i^a(x)\|} & \text{if } \|u_i^a(x)\| > m_i \\ u_i^a(x) & \text{otherwise} \end{cases} \quad (52)$$

Notice that this case is not covered by the theoretical results presented in the paper; however, we refer the reader to Ref. [7] for further analysis on the bounded input case.

Figure 7 shows the initial configuration of the agents. In Fig. 8, the agents are moving in straight lines toward their corresponding equilibria until they detect each other on the boundary of the safety regions. This is when the avoidance control inputs get activated in Eq. (47) and the agents begin to resolve the conflict, as seen in Figs. 9 and 10. Once the conflict has been resolved, the agents start heading toward their corresponding equilibrium points, as depicted in Fig. 11. The complete trajectories of the agents are shown in Fig. 12 with final conditions placed at $(-100,-100)$, $(-100,100)$, $(100,100)$, and $(100,-100)$, respectively.

**4.3 Example 3: Double Integrator Dynamics.** In order to illustrate the fact that the full state vector of each agent does not have to be solely comprised of the position of the agent, we consider the agents with second order dynamics given by

$$\ddot{x}_i = u_i \quad \forall\, i \in \mathbf{N} \quad (53)$$

The sensing and avoidance regions for all the agents are given by the similar functions as in the previous examples. It is easy to verify that the control laws

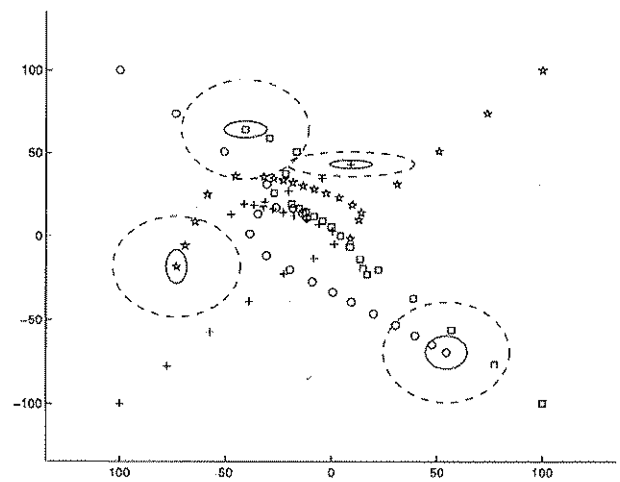


**Fig. 10 Conflict resolved**

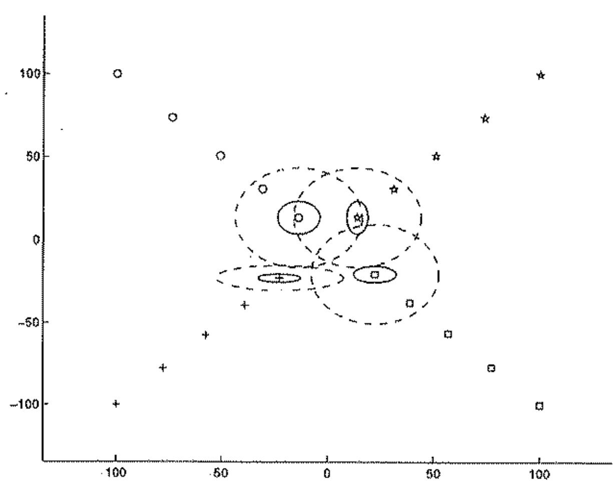


**Fig. 8 Agents start moving toward the equilibrium**

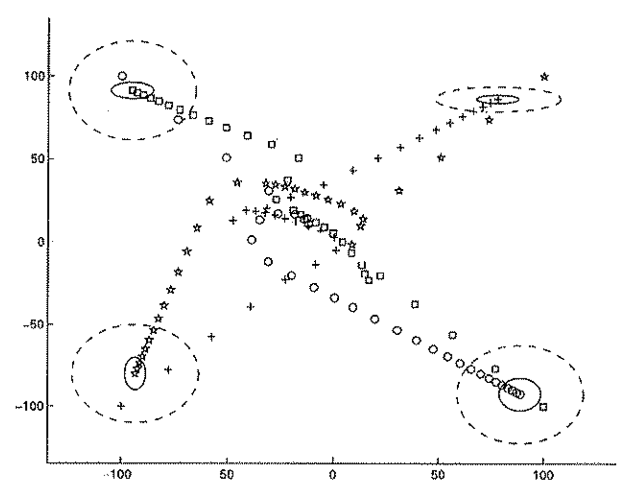


**Fig. 11 Agents continuing toward the equilibrium**

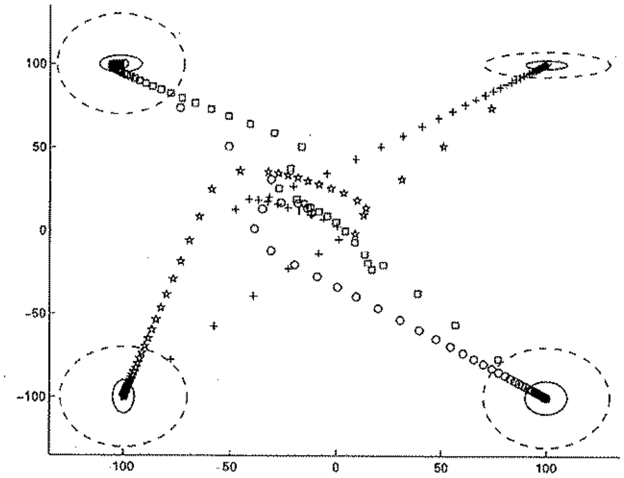


**Fig. 12 Initial to final configuration**

$$u_i = -k_i(x_i - x_i^e) - b_i\dot{x}_i \quad \forall\, i \in \mathbf{N} \tag{54}$$

where $k_i$ and $b_i$ are positive constants, will drive the agents to their corresponding equilibrium points. For this example, we choose the weight matrices $Q_i = \left[\begin{smallmatrix} I & 0 \\ 0 & 2I \end{smallmatrix}\right]$ and $R_i = I$, which result in optimal values of $k_i^o = 1$ and $b_i^o = 2$. Then, we append the control law (54) with the avoidance control laws $u_i^a$.

Figure 13 shows the initial configuration of the agents. In Fig. 14, the agents are moving in straight lines toward their corresponding equilibria until they start sensing each other. This is when the avoidance control laws get activated and the agents start resolving the conflict, as seen in Figs. 15 and 16. Once the conflict has been resolved the agents start to head towards their corresponding equilibria, as shown in Fig. 17. The complete trajectories of the agents are shown in Fig. 18 with final conditions placed at (−100,−100), (−100,100), (100,100), and (100,−100), respectively.

## 5 Conclusion

In this paper, a design of cooperative control laws based on the idea of avoidance control introduced in Ref. [1] has been applied to multiagent systems. The methodology is easy to implement, guarantees collision-free conflict resolution, and may be appended to already designed optimal control laws of independent agents.

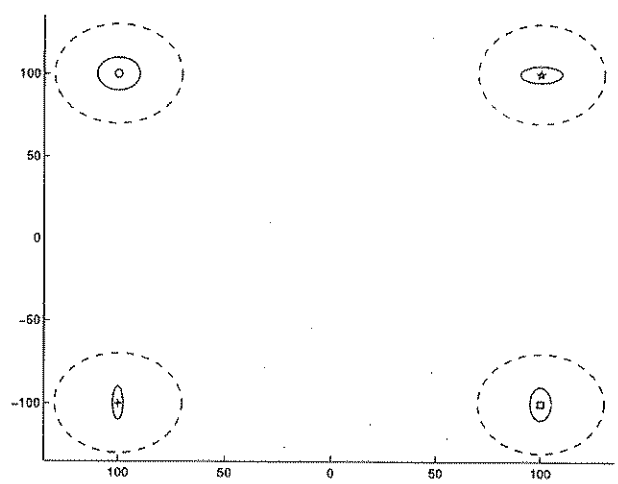


**Fig. 13 Initial configuration**

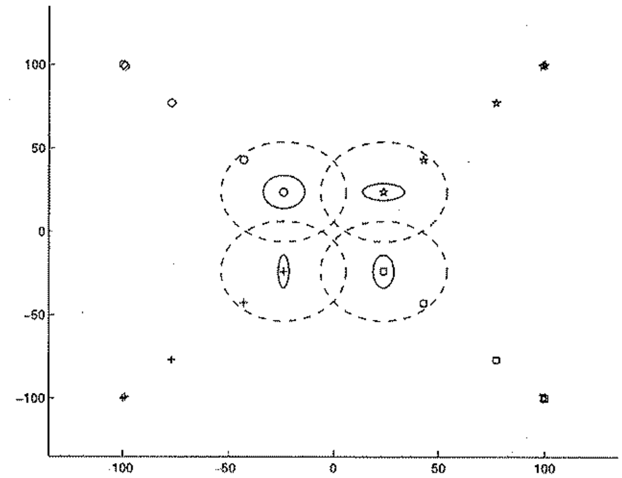


**Fig. 14 Agents start moving toward the equilibrium**

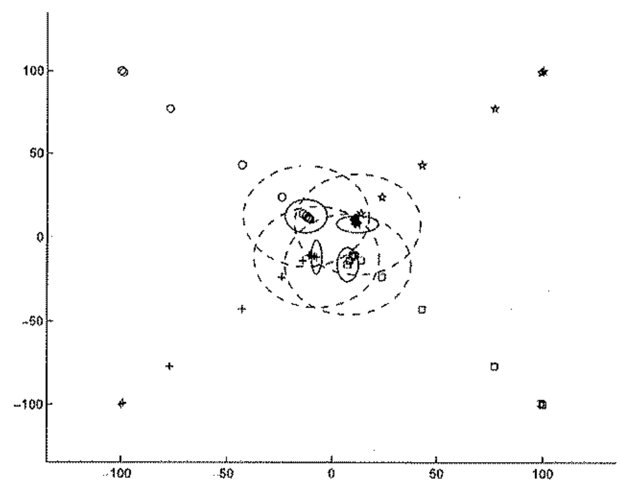


**Fig. 15 Agents starting to resolve the conflict**

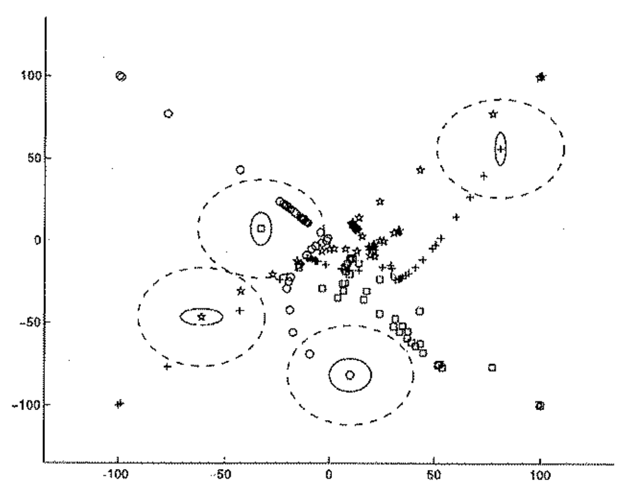


**Fig. 16 Conflict resolved**

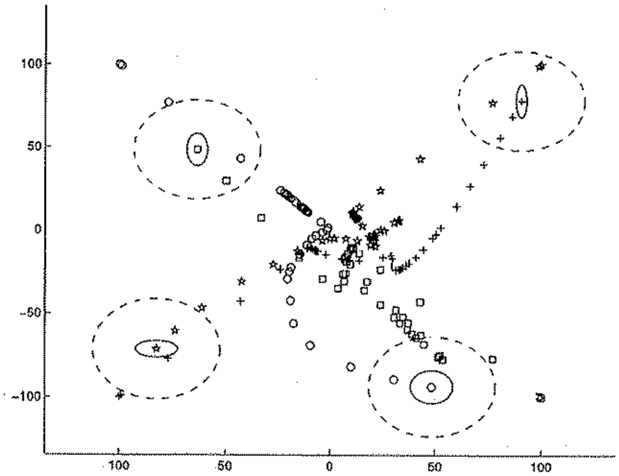


Fig. 17 Agents continuing toward the equilibrium

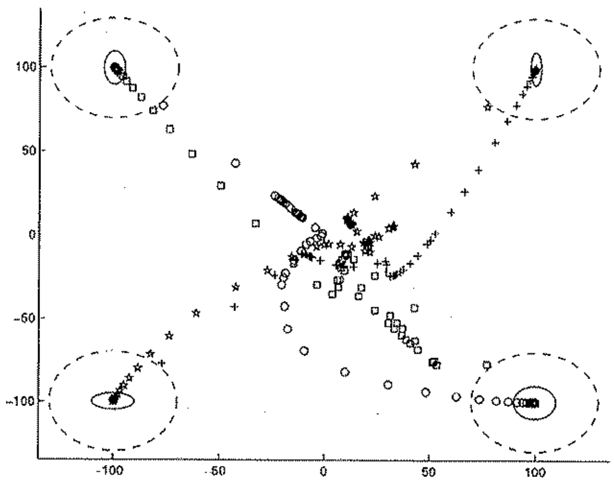


Fig. 18 Initial to final configuration

Our future research directions include the noncooperative case where some of the agents are not fully cooperating to avoid collisions [7] or the agents' detection sensors are not fully reliable, and the possibility of using avoidance value functions other than rational (e.g., logarithm functions). In addition, we plan to consider a variety of different feedback controllers such as dynamic, adaptive, and output types of controllers. Another important issue is to study the correlation between the shape and the size of both the avoidance and safety regions and dynamic capabilities of the agents described by their dynamic models.

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