Design of Robust Static Output Feedback for Large-Scale Systems

A. I. Zečević and D. D. Šiljak

***Abstract*—The design of static output feedback is of fundamental importance in control theory. In this note, we propose a new approach to this problem, based on linear matrix inequalities. A distinguishing feature of the method is its ability to handle large-scale problems with additive nonlinearities. The resulting control is robust with respect to uncertainties, and can incorporate several types of information structure constraints. The effectiveness of the proposed strategy is demonstrated by application to a practical large-scale system.**

***Index Terms*—Large-scale systems, linear matrix inequalities (LMIs), nonlinear systems, robust control, static output feedback.**

I. Introduction

The design of static output feedback is one of the central problems in control theory [1]–[6]. Given a linear, time-invariant system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned} \tag{1}$$

where $x \in R^n$ is the state of the system, $u \in R^m$ is the input vector and $y \in R^q$ represents the output, the objective is to determine a control law

$$u = Ky \tag{2}$$

that results in a stable closed-loop matrix $A+BKC$. As is well known, in the single-input–single-output (SISO) case, an appropriate gain matrix K can be found using graphical techniques such as root locus or the Nyquist method. The multivariable case, on the other hand, remains elusive, and no analytic solution is currently available.

A number of numerical techniques have been proposed to address the multivariable problem, including a variety of iterative schemes [7]–[9], trust region methods [10] and semidefinite programming algorithms [11]. It should be noted, however, that all of these methods are primarily concerned with linear time-invariant systems, and devote little attention to nonlinearities and issues related to large-scale problems (such as computational complexity and information structure constraints). With that in mind, the main objective of this note will be to formulate the design of static output feedback in the framework of linear matrix inequalities (LMIs) [12]–[14]. Two features of the proposed approach deserve particular attention.

i) Unlike most of the existing strategies, our method is designed to produce robust output feedback for an important class of *nonlinear* systems. Models with this type of nonlinearity arise in a wide variety of applications, ranging from electric power systems to aerospace design and vehicle control (e.g., [15] and [16]).
ii) The proposed LMI formulation is suitable for large-scale systems, where computational complexity and information structure constraints are key considerations. This approach can accommodate several types of gain matrix structures, and the resulting control laws can be easily implemented in a multiprocessor environment.

The note is organized as follows. In Section II, we formulate the design of static output feedback as an LMI optimization problem, and identify a set of sufficient conditions for the existence of a solution. The method is then extended to include decentralized information structure constraints, which arise in many practical large-scale systems. Section III describes generalizations to bordered block diagonal (BBD) gain matrix structures, and discusses the implementation of such a control in a multiprocessor environment. A large-scale example is provided to demonstrate the effectiveness of this approach.

II. Design of Static Output Feedback Using LMIs

Let us consider a nonlinear system described by the differential equations

$$\begin{aligned}\dot{x} &= Ax + h(x) + Bu\\ y &= Cx\end{aligned} \tag{3}$$

where A, B, and C are constant $n \times n$, $n \times m$ and $q \times n$ matrices, and $h: R^n \rightarrow R^n$ is a piecewise-continuous nonlinear function in x, satisfying $h(0) = 0$. The term $h(x)$ may contain uncertainties, but we assume that it can be bounded by a quadratic inequality

$$h^T(x)h(x) \leq \alpha^2 x^T H^T H x \tag{4}$$

where H is a constant matrix, and $\alpha > 0$ is a scalar parameter.

If we apply a state feedback control law $u = Kx$, the global asymptotic stability of the closed-loop system can be established using a Lyapunov function

$$V(x) = x^T P x \tag{5}$$

where P is a symmetric positive definite matrix. Sufficient conditions for stability are well known, and can be expressed as a pair of inequalities

$$\begin{gathered} x^T P x > 0\\ \begin{bmatrix} x\\ h\end{bmatrix}^T \begin{bmatrix} (A+BK)^T P + P(A+BK) & P\\ P & 0\end{bmatrix} \begin{bmatrix} x\\ h\end{bmatrix} < 0.\end{gathered} \tag{6}$$

Defining $Y = \tau P^{-1}$ (where τ is a positive scalar), $L = KY$, and $\gamma = 1/\alpha^2$, the control design can now be formulated as an LMI problem in γ, κ_Y, κ_L, Y and L [17], [18].

Problem 1: Minimize $a_1\gamma + a_2\kappa_Y + a_3\kappa_L$ subject to

$$Y > 0 \tag{7}$$

$$\begin{bmatrix} AY + YA^T + BL + L^T B^T & I & YH^T\\ I & -I & 0\\ HY & 0 & -\gamma I\end{bmatrix} < 0 \tag{8}$$

$$\gamma - \frac{1}{\bar{\alpha}^2} < 0 \tag{9}$$

and

$$\begin{bmatrix} -\kappa_L I & L^T\\ L & -I\end{bmatrix} < 0; \quad \begin{bmatrix} Y & I\\ I & \kappa_Y I\end{bmatrix} > 0. \tag{10}$$

Several comments need to be made regarding this design procedure.

Remark 1: The gain matrix is computed as $K = LY^{-1}$, and its norm is implicitly constrained by inequalities (10), which imply that

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$\|K\| \leq \sqrt{\kappa_L}\kappa_Y$. This is necessary in order to prevent unacceptably high gains that an unconstrained optimization may otherwise produce [17], [18].

Remark 2: If the optimization problem (7)–(10) is feasible, the resulting gain matrix stabilizes the closed-loop system for *all* nonlinearities satisfying (4). Condition (9) additionally secures that α is greater than some desired robustness bound $\bar{\alpha}$.

For the purposes of static output feedback, the closed-loop system must have the form

$$\dot{x} = (A + BKC)x + h(x). \tag{11}$$

With that in mind, our main objective in the following will be to modify the LMI optimization (7)–(10) so that the product LY^{-1} can be factorized as

$$LY^{-1} = KC. \tag{12}$$

A simple approach to this problem would be to look for a solution in which matrix L has the form

$$L = L_C U^T \tag{13}$$

where U is a fixed $n \times q$ matrix and L_C is an unknown $m \times q$ matrix. In that case, (12) holds whenever

$$U^T Y^{-1} = C \tag{14}$$

with $K = L_C$.

Condition (14) can be included in the LMI optimization by adding the equality constraint

$$YC^T = U. \tag{15}$$

In this context, we should note that (15) is *automatically* satisfied if we set $U = C^T$, and look for Y in the form

$$Y = QY_Q Q^T + C^T (CC^T)^{-1} C \tag{16}$$

where Q is an $n \times (n-q)$ matrix such that

$$Q^T C^T = 0 \tag{17}$$

and Y_Q is an unknown symmetric matrix of dimension $(n-q) \times (n-q)$. Under such circumstances, the LMI optimization does *not* require an explicit equality constraint, but the number of LMI variables associated with Y is reduced from $n(n+1)/2$ to $(n-q)(n-q+1)/2$.

The reduction of variables due to (16) can have a detrimental effect on the feasibility of the optimization, particularly in cases where q is relatively large. For that reason, we propose to introduce additional LMI variables by looking for a solution of Problem 1 in the form

$$\begin{aligned} Y &= Y_0 + UY_C U^T \\ L &= L_C U^T \end{aligned} \tag{18}$$

where Y_0 and Y_C are unknown symmetric matrices of dimensions $n \times n$ and $q \times q$, respectively. For any given choice of U, the optimization (7)–(10) then becomes an LMI problem in $\gamma, \kappa_Y, \kappa_L, Y_0, Y_C$ and L_C.

To see the connection between (18) and the desired output feedback structure, we should observe that Y^{-1} can be expressed using the Sherman–Morrison formula as (e.g., [19])

$$Y^{-1} = Y_0^{-1} - SRU^T Y_0^{-1} \tag{19}$$

where

$$\begin{aligned} S &= Y_0^{-1} U Y_C \\ R &= [I + U^T S]^{-1}. \end{aligned} \tag{20}$$

It is now easily verified that condition

$$Y_0 C^T = U \tag{21}$$

ensures that $LY^{-1} = KC$, with

$$K = L_C (I - U^T SR). \tag{22}$$

As before, the equality constraint (21) can be *automatically* satisfied if we set $U = C^T$, and look for Y_0 in the form

$$Y_0 = QY_Q Q^T + C^T (CC^T)^{-1} C \tag{23}$$

where Q and Y_Q have the same properties as in (16) and (17). We should note, however, that the overall number of LMI variables associated with Y is now increased by $q(q+1)/2$, due to the presence of matrix Y_C. The corresponding design procedure (referred to in the following as Algorithm 1) is a relatively simple modification of Problem 1, and can be summarized as follows.

Algorithm 1. *Control Design with* $U = C^T$

Step 1) Compute an $n \times (n-q)$ matrix Q of full rank that satisfies (17).

Step 2) Set $U = C^T$, and solve optimization Problem 1 for γ, κ_Y, κ_L, Y_Q, Y_C and L_C, with

$$\begin{aligned} Y &= QY_Q Q^T + C^T (CC^T)^{-1} C + C^T Y_C C \\ L &= L_C C. \end{aligned} \tag{24}$$

Step 3) Compute the gain matrix K using (20) and (22).

Although Algorithm 1 is quite straightforward, our numerical experiments suggest that the number of nonzero elements in C can affect the feasibility of the optimization. As a result, it is reasonable to consider a more general scenario, in which $U \neq C^T$. In order to do that, we first need to establish conditions under which (21) has a symmetric solution. The following theorem provides some insight into this problem (the proof can be found in [20]).

Theorem 1: If (3) has more than one output, (21) is *singular*, and a symmetric solution Y_0 exists only for special choices of U.

In view of Theorem 1, we now propose an alternative design strategy which allows us to choose $U \neq C^T$.

Algorithm 2. *Control Design with* $U \neq C^T$

Step 1) Arrange the upper triangular elements of Y_0 into a vector y_0, and rewrite (21) as

$$Gy_0 = u \tag{25}$$

where G is contructed using matrix C and u is formed from the columns of matrix U.

Step 2) Perform a QR factorization of matrix G, and express (25) as

$$\tilde{Q}\tilde{R}Ey_0 = u \tag{26}$$

where $\tilde{Q}$ is an orthogonal $qn \times qn$ matrix, $\tilde{R}$ is upper triangular and E is a permutation matrix. If G has rank deficiency ρ, matrix E secures that the last ρ rows of $\tilde{R}$ are zero.

Step 3) Construct a matrix U such that the last ρ rows of $\tilde{Q}^T u$ are zero. Equation (25) is then guaranteed to be consistent.

Step 4) Denoting $\mathrm{rank}(\tilde{R}) = l$, partition the permuted vector $\tilde{y}_0 = Ey_0$ as $[\tilde{y}_1^T \; \tilde{y}_2^T]^T$, where $\tilde{y}_1$ represents the first l elements. The solution of (26) must then satisfy

$$\tilde{y}_1 = \Gamma_0 + \Gamma_1 \tilde{y}_2 \tag{27}$$

where Γ_0 and Γ_1 are constant matrices defined by $\tilde{Q}$, $\tilde{R}$ and U.

Step 5) Equation (27) implies that not all elements of Y_0 are independent, and that this matrix must have the form

$$Y_0 = \Pi_0 + \sum_{i=1}^{s} \beta_i \Pi_i. \tag{28}$$

In (28), β_i $(i = 1, \ldots, s)$ represent the elements of Y_0 that correspond to $\tilde{y}_2$, and Π_i $(i = 0, 1, \ldots, s)$ are constant symmetric matrices that can easily be computed from Γ_0 and Γ_1.

Step 6) Solve optimization Problem 1 for γ, κ_Y, κ_L, Y_C, $\beta_1, \ldots, \beta_s$ and L_C, with

$$\begin{aligned} Y &= \Pi_0 + \sum_{i=1}^{s} \beta_i \Pi_i + U Y_C U^T \\ L &= L_C U^T. \end{aligned} \tag{29}$$

III. Application to Large-Scale Systems

Let us now consider (3) with the added assumption that matrices $B_D = \mathrm{diag}\{B_1, \ldots, B_N\}$ and $C_D = \mathrm{diag}\{C_1, \ldots, C_N\}$ consist of $n_i \times m_i$ and $q_i \times n_i$ diagonal blocks, respectively. If we partition matrix A in accordance with the blocks of B_D, (3) can obviously be represented as N interconnected subsystems

$$\begin{aligned} \dot{x}_i &= A_{ii} x_i + \sum_{j=1}^{N} A_{ij} x_j + h_i(x) + B_i u_i \\ y_i &= C_i x_i. \end{aligned} \tag{30}$$

Since such a system is both input and output decentralized, it is natural to consider a control law of the form (e.g., [16])

$$u_i = K_i y_i, \qquad (i = 1, 2, \ldots, N). \tag{31}$$

In order to obtain this type of output feedback structure, the following additional requirements need to be incorporated into Problem 1.

Requirement 1: Matrix Y must have the form

$$Y = Y_0 + U_D Y_C U_D^T \tag{32}$$

where Y_0 and Y_C are unknown $n \times n$ and $q \times q$ block diagonal matrices, respectively. The diagonal blocks of Y_0 have dimension $n_i \times n_i$, and those of Y_C have dimension $q_i \times q_i$.

Requirement 2: Matrix U_D is a user-defined block diagonal matrix of dimension $n \times q$, consisting of $n_i \times q_i$ blocks.

Requirement 3: Matrix Y_0 must satisfy the equality constraint

$$Y_0 C_D^T = U_D. \tag{33}$$

Note that if U_D is chosen as $U_D = C_D^T$, this condition is automatically satisfied by any matrix Y_0 of the form

$$Y_0 = Q_D Y_Q Q_D^T + C_D^T \left(C_D C_D^T\right)^{-1} C_D \tag{34}$$

where Q_D is an $n \times (n - q)$ block diagonal matrix such that

$$Q_D^T C_D^T = 0. \tag{35}$$

In that case, we need to compute an $(n-q) \times (n-q)$ matrix Y_Q, which is symmetric and block diagonal, with $(n_i - q_i) \times (n_i - q_i)$ blocks. We should also point out that a more general solution to this problem can be developed along the lines of Algorithm 2 and Theorem 1.

Requirement 4: Matrix L must have the form

$$L = L_C U_D^T \tag{36}$$

where L_C is a block diagonal $m \times q$ matrix with blocks of dimension $m_i \times q_i$.

As in the previous section, it is easily verified that Requirement 3) implies $LY^{-1} = K_D C_D$, with

$$K_D = L_C \left(I - U_D^T S R\right). \tag{37}$$

Requirements 1), 2), and 4) also ensure that $K_D = \mathrm{diag}\{K_1, \ldots, K_N\}$ is a block diagonal matrix, with blocks K_i of dimension $m_i \times q_i$. The resulting decentralized output control is clearly desirable in the case of large-scale systems, where implementation and information structure constraints are crucial factors. This approach also has a decided computational advantage, since the block diagonal structure of matrices Y_0 and L_C results in a substantial reduction in the overall number of LMI variables.

A generalization of decentralized output control is a feedback law of the form

$$\begin{aligned} u_i &= K_{ii} y_i + K_{iN} y_N, \qquad (i = 1, 2, \ldots, N-1) \\ u_N &= \sum_{i=1}^{N} K_{Ni} y_i \end{aligned} \tag{38}$$

which corresponds to an $m \times q$ BBD gain matrix

$$K_{\mathrm{BBD}} = \begin{bmatrix} K_{11} & 0 & \ldots & K_{1N} \\ 0 & K_{22} & \ldots & K_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ K_{N1} & K_{N2} & \ldots & K_{NN} \end{bmatrix}. \tag{39}$$

Control laws of this type can be obtained in the same way as decentralized output feedback, the only difference being that matrix L_C in Requirement 4) must now be a BBD matrix of dimension $m \times q$ with a block structure that is identical to (39). This follows directly from the fact that $I - U_D^T S R$ is a block diagonal matrix with $q_i \times q_i$ diagonal blocks, by virtue of Requirements 1)–3).

The following remarks need to be made regarding the application of BBD output control to large-scale systems.

Remark 3: Given matrices $\{A, B, C\}$ where A is large and sparse, it is always possible to simultaneously permute A into the BBD form and secure a compatible block-diagonal structure for B and C. A graph-theoretic algorithm for such a permutation was developed in [20], following the ideas introduced in [21].

Remark 4: In order for the proposed BBD design to be feasible for large-scale systems, it is necessary to ensure that the number of LMI variable associated with Y_Q and L_C is not excessively large. In the

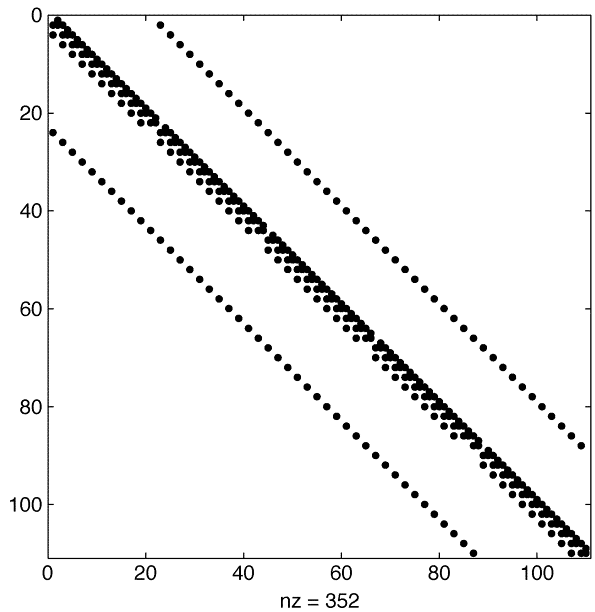

Fig. 1. Nonzero structure of the system matrix A.

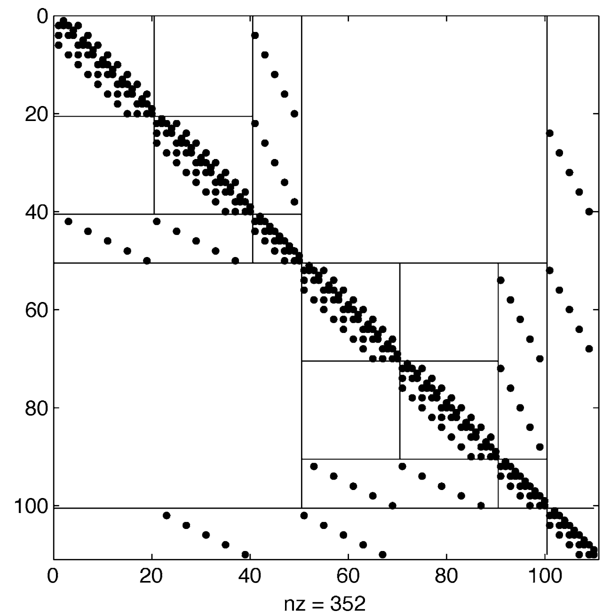

Fig. 2. System matrix $A_{\rm BBD}$ after a BBD decomposition.

case of matrix Y_Q, this can be accomplished by looking for a solution in which the ith diagonal block has the form

$$Y_Q^{(i)} = D_i + N_i \bar{Y}_Q^{(i)} N_i^T \tag{40}$$

where D_i is an unknown $(n_i - q_i) \times (n_i - q_i)$ diagonal matrix (this term is necessary to secure that $Y_Q^{(i)}$ is nonsingular), N_i is a fixed matrix of dimension $(n_i - q_i) \times r_i$, and $\bar{Y}_Q^{(i)}$ is an unknown $r_i \times r_i$ matrix. If $r_i \ll n_i$, the computational savings can be substantial.

The number of variables associated with L_C can be reduced along similar lines. Recalling that this matrix has a BBD structure with blocks L_{ik} of dimensions $m_i \times q_k$, the total number of variables associated with L_C is

$$\eta(L_C) = \sum_{i=1}^{N} m_i q_i + m_N \sum_{i=1}^{N-1} q_i + q_N \sum_{i=1}^{N-1} m_i. \tag{41}$$

Note, however, that such a matrix can always be expressed as $L_C \equiv L_D M$, where L_D consists of $m_i \times r_i$ diagonal blocks only, and M has a BBD structure with blocks M_{ik} of dimension $r_i \times q_k$. With that in mind, we propose to fix matrix M and treat only L_D as a variable matrix in the LMI optimization. Since we can choose r_i freely, this allows us to significantly reduce the number of variables.

Remark 5: The implementation of BBD control in a multiprocessor environment is quite straightforward, since a nested BBD matrix structure is easily mapped onto a tree type parallel architecture. In such a scheme, the only communication tasks are single node gather and scatter operations, which results in low overhead (e.g., [22]).

The effectiveness of BBD output control is illustrated by the following example, in which we apply the proposed strategy for reducing computational complexity.

Example 1: Let us consider a mechanical system which consists of a two-dimensional array of elastically connected masses. For each mass the horizontal and vertical dynamics can be decoupled, so it is sufficient to consider only movement along the x—axis. The equations of motion for the ith mass are

$$\begin{aligned} \Delta \dot{x}_i &= v_{xi} \\ M_i \dot{v}_{xi} &= -\delta_i v_{xi} - \left(\sum_{j \neq i} \mu_{ij} \right) \Delta x_i + \sum_{j \neq i} \mu_{ij} \Delta x_j \end{aligned} \tag{42}$$

where Δx_i represents the horizontal displacement, δ_i is the damping and μ_{ij} is the elastic force acting between masses i and j. In the following, we will consider a 5 × 11 array with 14 inputs and 14 outputs, which corresponds to the 110 × 110 matrix A shown in Fig. 1. Following the algorithm described in [20], this matrix can be decomposed into the nested BBD form shown in Fig. 2 (matrices B and C can be simultaneously permuted into a compatible block-diagonal structure, as noted in Remark 3).

In our experiments, we assumed that 20% of the masses have no damping, and our objective was to design a BBD output control such that the real part of each closed-loop mode is less than −0.03. The structure of the decomposed system suggests three hierarchial levels. With that in mind, it is appropriate to partition the state vector as $[x_1^T \; x_2^T \; \ldots \; x_7^T]^T$ (in accordance with the diagonal blocks of the permuted matrix $A_{\rm BBD}$), and utilize seven processors connected in a tree-type architecture.

Regarding the computational complexity of the LMI optimization, we should note that the original problem involves some 2,000 variables. However, if we assume the form (40) with $r_i = 10$ for the four 20 × 20 blocks of Y_Q (while keeping the 10 × 10 blocks intact), and form matrix L_D using 2 × 6 diagonal blocks, the number of LMI variables reduces to 620. The problem was successfully solved under these assumptions. We also found that for this type of problem the extent of the reduction does not seem to be significantly influenced by the specific choice of matrices N_i and M. Indeed, the optimization was found to be feasible with the same number of LMI variables for several random choices of these matrices.

IV. Conclusion

In this note, we proposed a new approach for the design of static output feedback. Linear matrix inequalities were used to formulate the design as a convex optimization problem, which can be solved efficiently using standard numerical techniques. Such an approach can incorporate decentralized and BBD information structure constraints, and is applicable to an important class of large nonlinear systems. The effectiveness of the proposed strategy was demonstrated by application to a large-scale mechanical problem.

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A Direct Algebraic Approach to Observer Design Under Switching Measurement Equations

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***Abstract*—Based on the algebraic transformation of a switched linear measurement equation into a nonlinear, yet deterministic, equation, an asymptotic state observer is constructed for discrete-time linear systems whose observations are generated according to randomly switching measurement modes. The observer, which combines the algebraic transformation with a Newton observer applied to the resulting nonlinear measurement equation, is shown to be locally exponentially convergent under arbitrary mode sequences.**

***Index Terms*—Observer design, sensor failures, switched systems.**

I. INTRODUCTION

The emergence of increasingly complex engineering systems has triggered an intense focus on novel control-theoretic areas of research, including sensor and actuator networks, decentralized control, and fault-tolerant control. In order for such complex systems to behave in a satisfactory manner, i.e., to be subjected to effective control strategies, it is vitally important that the measured sensory data be incorporated in the control loop under various forms of unreliability. In particular, in a number of applications, including manufacturing, telecommunications, and embedded systems, sensor failures occur intermittently and go undetected, while only a finite number of possible sensory modes of operation exist, and are known. In other words, even though it is unknown which mode of operation the sensors obey to at any given time instant, a characterization of all possible sensory modes is assumed available *a priori*. In this note, we consider the particular class of discrete-time linear dynamical systems with randomly switching measurement equations, and we propose a local exponential state observer for such systems.

In other words, we consider the single-output autonomous system

$$\begin{aligned} x_{k+1} &= Ax_k \\ y_k &= C(\theta_k)x_k \end{aligned} \tag{1}$$

where x_k and y_k are in $\mathbb{R}^n$ and $\mathbb{R}$, respectively, where the *mode* θ_k takes values in $\{1, \dots, m\}$, and where $A, C(1), \dots, C(m)$ are constant matrices of compatible dimensions. We assume that the mode sequence $\{\theta_k\}_{k=0}^{\infty}$ is arbitrary, indexing the measurement equation in such a way that $C(\theta_k)$ switches randomly among $C(1), \dots, C(m)$, modeling the m different sensory modes. Throughout the note, we will further assume that A is invertible, which is a natural assumption for sampled linear systems. In fact, it is easily shown (e.g., [6]) that sampling a continuous-time linear system with *arbitrarily* switched measurement equations actually results in a switched linear system (1), which is not

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