Inclusion Principle for Descriptor Systems

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***Abstract*—The purpose of this paper is to propose an expansion-contraction framework for linear constant descriptor systems within the inclusion principle for dynamic systems. Our primary objective is to provide an explicit characterization of the expansion process whereby a given descriptor system is expanded into the larger space where all its solutions are reproducible by the expanded descriptor system if appropriate initial conditions are selected. When a control law is formulated in the expanded space, the proposed characterizations provide contractibility conditions for implementation of the control law in the original system. A full freedom is provided for selecting appropriate matrices in the proposed expansion-contraction control scheme. In particular, the derived theoretical framework serves as a flexible environment for expansion-contraction control design of descriptor systems under overlapping information structure constraints.**

***Index Terms*—Contractibility, contractions, controllability at infinity, descriptor systems, expansions, inclusion principle.**

I. Introduction

In recent years, the inclusion principle for dynamic systems [1], [2] has emerged as a flexible and powerful mathematical framework for comparing properties and performance of systems with different dimensions. The principle has found applications in a wide variety of theoretical and practical situations involving model-reduction, large dynamic systems, optimal control, parallel computations, inclusions of dynamic controllers and observers, expert systems and decentralized control of hybrid, mechanical and electrical systems, control of segmented telescope and platoons of vehicles in the air and on the ground, as well as the analysis of the finite word-length [3]–[24]. Recently, the development of the inclusion principle for continuous, discrete and stochastic systems [11]–[21] has been focused on formulating conditions for expansions and contractions of control systems, which can help resolve outstanding theoretical and practical aspects of the principle in building control systems under overlapping information structure constraints.

Descriptor systems, which are also called singular systems or generalized state space systems, appear as models in as diverse areas as electrical circuits and multibody systems, chemical engineering and economic systems, mechanical structure and biological systems [25]–[34]. Motivated by this wide-spread use of descriptor models, we formulate in this paper the inclusion principle for expansion and contraction of descriptor systems including contractibility of control laws. In deriving explicit algebraic characterizations of the expansion-contraction process, we shall define canonical forms for descriptor systems within the inclusion framework, which generalize canonical forms obtained for linear time-invariant systems [22]. Theoretical results established in this work reveal that the expansion-contraction and contractibility in the inclusion principle for descriptor systems are much more complicated than those for standard linear time-invariant systems. In particular, the expanded system cannot be generally expressed in terms of the original system. Furthermore, a control law for the expanded system, which is contractible to a control law for the original systems, cannot be always expressed in terms of the given control law for the original system. These difficulties can be overcome only if either the original system is controllable at infinity, or the set of the order of the poles of the expanded system at infinity contains the set of the order of the poles of the original system at infinity.

The paper is organized as follows. In the next section, we derive the inclusion principle for descriptor systems. Section III contains our main results. Some concluding remarks are provided in Section IV. Proofs of a number of lemmas and theorems are given in Appendix.

II. Inclusion and Contractibility

Consider a pair of descriptor systems

$$\mathbf{S}: \quad \begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) & x(0) = x^0 \\ y(t) = Cx(t) \end{cases} \tag{1}$$

and

$$\tilde{\mathbf{S}}: \quad \begin{cases} \tilde{E}\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t), & \tilde{x}(0) = \tilde{x}^0 \\ \tilde{y}(t) = \tilde{C}\tilde{x}(t) \end{cases} \tag{2}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^l$ are the state, input and output of system $\mathbf{S}$ at time $t \geq 0$, and $\tilde{x}(t) \in \mathbf{R}^{\tilde{n}}$, $\tilde{u}(t) \in \mathbf{R}^{\tilde{m}}$, $\tilde{y}(t) \in \mathbf{R}^{\tilde{l}}$ are those of $\tilde{\mathbf{S}}$, and $E, A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{l\times n}$, $\tilde{E}, \tilde{A} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$, $\tilde{B} \in \mathbf{R}^{\tilde{n}\times\tilde{m}}$, $\tilde{C} \in \mathbf{R}^{\tilde{l}\times\tilde{n}}$ are constant matrices. For descriptor systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$, unique solutions are guaranteed to exist if and only if the pencils $sE - A$ and $s\tilde{E} - \tilde{A}$ are regular, i.e., $\det(sE - A)$ and $\det(s\tilde{E} - \tilde{A})$ do not vanish identically. For regular systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$, in order to have standard continuous solutions, the inputs $u(t)$ and $\tilde{u}(t)$ should be sufficiently smooth, that is, $u(t)$ and $\tilde{u}(t)$ must belong to some suitable function spaces [34], [41], say, $\mathcal{U}_{ad}$ and $\tilde{\mathcal{U}}_{ad}$, respectively. Otherwise, if $u(t)$ and $\tilde{u}(t)$ are not sufficiently smooth, then the impulses may arise in the responses of systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ even if these two systems are regular. For this reason, descriptor systems are considerably more difficult to analyze and control than the standard linear time-invariant systems.

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Suppose the pencils $sE - A$ and $s\tilde{E} - \tilde{A}$ are regular, and

$$n \le \tilde{n}, \quad m \le \tilde{m}, \quad l \le \tilde{l},$$

that is, $\mathbf{S}$ is smaller than $\tilde{\mathbf{S}}$. The initial states x^0 and $\tilde{x}^0$ for systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ with the given inputs $u(t) \in \mathcal{U}_{ad}$ and $\tilde{u}(t) \in \tilde{\mathcal{U}}_{ad}$ are said to be consistent, if there exist $x(t)$ and $\tilde{x}(t)$ satisfying (1) and (2), respectively. Denote $x(t; x^0, u)$ and $y[x(t; x^0, u)]$ to represent the state behavior and the corresponding output of system $\mathbf{S}$ for a fixed input $u(t) \in \mathcal{U}_{ad}$ and initial state $x(0) = x^0$. Similar notation $\tilde{x}(t; \tilde{x}^0, \tilde{u})$ and $\tilde{y}[\tilde{x}(t; \tilde{x}^0, \tilde{u})]$ is used for the state behavior and output of system $\tilde{\mathbf{S}}$.

Let us link systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ through the following transformations:

$$V : \mathbf{R}^n \longrightarrow \mathbf{R}^{\tilde{n}}, \quad L : \mathbf{R}^m \longrightarrow \mathbf{R}^{\tilde{m}}, \quad T : \mathbf{R}^l \longrightarrow \mathbf{R}^{\tilde{l}} \quad (3)$$

where

$$\mathrm{r}(V) = n, \quad \mathrm{r}(L) = m, \quad \mathrm{r}(T) = l \quad (4)$$

and $\mathrm{r}(V)$, $\mathrm{r}(L)$ and $\mathrm{r}(T)$ denote the rank of V, L and T, respectively.

Denote the unique pseudoinverses of V, L and T by $V^{(+)}$, $L^{(+)}$ and $T^{(+)}$, respectively, and recall the definition of the Inclusion Principle [1], [2]:

Definition 1: The system $\tilde{\mathbf{S}}$ includes the system $\mathbf{S}$, that is, $\mathbf{S}$ is included by $\tilde{\mathbf{S}}$, if there exists a triplet (V, L, T) satisfying (3) and (4) such that, for any fixed $u(t)$ and any initial state x^0, the consistency conditions

$$\tilde{x}^0 = Vx^0, \quad \tilde{u}(t) = Lu(t), \quad \forall t \ge 0 \quad (5)$$

imply

$$\begin{cases} x(t; x^0, u) = V^{(+)}\tilde{x}(t; \tilde{x}^0, \tilde{u}) & \forall t \ge 0 \\ y\left[x(t; x^0, u)\right] = T^{(+)}\tilde{y}\left[\tilde{x}(t; \tilde{x}^0, \tilde{u})\right] & \forall t \ge 0. \end{cases} \quad (6)$$

If the system $\tilde{\mathbf{S}}$ includes the system $\mathbf{S}$, then system $\tilde{\mathbf{S}}$ is said to be an *expansion* of the system $\mathbf{S}$ and system $\mathbf{S}$ is a *contraction* of system $\tilde{\mathbf{S}}$.

The inclusion principle has been used to *expand* overlapping decentralized control laws into a larger space, where they appear disjoint, design disjoint laws by known methods, and *contract* them to the original space for implementation (e.g., [2]). The central issue in the expansion-contraction process is the problem of *contractibility* [1], [4], [11], [12], [18], [19]. In this paper we consider the expansion-contraction of state feedback for descriptor systems, leaving the more intricate case of output feedback for future research. We state the following:

Definition 2: The control law

$$\tilde{u} = -\tilde{K}\tilde{x} + \tilde{v}$$

for system $\tilde{\mathbf{S}}$, where $\tilde{K} \in \mathbf{R}^{\tilde{m} \times \tilde{n}}$ is a constant gain matrix and $\tilde{v} \in \mathbf{R}^{\tilde{m}}$ is a reference input, is *contractible* to the control law

$$u = -Kx + v$$

for implementation in system $\mathbf{S}$, where $K \in \mathbf{R}^{m \times n}$ is a constant gain matrix and $v \in \mathbf{R}^m$ is a reference input, if there exists a triplet (V, L, T) satisfying (3) and (4) such that one of the following two statements holds:

a) The consistency conditions

$$\tilde{x}^0 = Vx^0, \quad \tilde{u}(t) = Lu(t)$$

imply

$$\begin{cases} x(t; x^0, u) = V^{(+)}\tilde{x}(t; \tilde{x}^0, \tilde{u}) \\ LKx(t; x^0, u) = \tilde{K}\tilde{x}(t; \tilde{x}^0, \tilde{u}) \end{cases} \quad (7)$$

for all $t \ge 0$, for any fixed input $u(t)$ and initial state x^0.

b) The consistency conditions

$$\tilde{x}^0 = Vx^0, \quad u(t) = L^{(+)}\tilde{u}(t)$$

imply

$$\begin{cases} x(t; x^0, u) = V^{(+)}\tilde{x}(t; \tilde{x}^0, \tilde{u}) \\ Kx(t; x^0, u) = L^{(+)}\tilde{K}\tilde{x}(t; \tilde{x}^0, \tilde{u}) \end{cases} \quad (8)$$

for all $t \ge 0$, for any fixed input $\tilde{u}(t)$ of system $\tilde{\mathbf{S}}$ and initial state x^0.

It should be pointed out that both conditions in (a) and (b) above ensure that the closed-loop system

$$\tilde{E}\dot{\tilde{x}} = (\tilde{A} + \tilde{B}\tilde{K})\tilde{x} + \tilde{B}\tilde{v}$$

includes the closed-loop system

$$E\dot{x} = (A + BK)x + Bv.$$

This property plays an important role in the application of the inclusion principle to overlapping decentralized control.

In the next section, we will introduce the expansion-contraction scheme and contractibility conditions in the inclusion principle for descriptor systems. Our results will provide a systematic procedure for choosing matrices $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{C}, \tilde{K})$ of the system $\tilde{\mathbf{S}}$ in designing controllers for systems under overlapping information structure constraints.

III. Main Results

The purpose of this section is twofold. First, we derive explicit algebraic characterization of the inclusion framework for descriptor systems and, second, we use the framework to obtain the same type of characterizations for contractibility of feedback control laws. The characterizations will parameterize all systems $\tilde{\mathbf{S}}$ which include $\mathbf{S}$, and all control laws $\tilde{u} = -\tilde{K}\tilde{x} + \tilde{v}$ for system $\tilde{\mathbf{S}}$ which are contractible to the control law $u = -Kx + v$ and can be implemented in system $\mathbf{S}$.

The following preliminary lemmas will be important in the developments below.

Lemma 1: [36], [37] Let $\mathcal{A} \in \mathbf{R}^{\mu \times \mu}$, $\mathcal{B} \in \mathbf{R}^{\mu \times \nu}$, $\mathcal{C} \in \mathbf{R}^{\tau \times \mu}$. Then, $\mathcal{C}\mathcal{A}^i\mathcal{B} = 0$, for $i = 0, 1, \cdots, \mu - 1$ if and only if $\max_{s \in \mathbf{C}} \mathrm{r}\begin{bmatrix} sI - \mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{bmatrix} = \mu$.

Lemma 2: [22] Let $\mathcal{A} \in \mathbf{R}^{\mu \times \mu}$, $\mathcal{B} \in \mathbf{R}^{\mu \times \nu}$, $\mathcal{C} \in \mathbf{R}^{\tau \times \mu}$ and $\mathcal{D} \in \mathbf{R}^{\tau \times \nu}$.

i) $\max_{s \in \mathbf{C}} \mathrm{r}\begin{bmatrix} sI - \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \mu$ if and only if $\mathcal{D} = 0$ and $\max_{s \in \mathbf{C}} \mathrm{r}\begin{bmatrix} sI - \mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{bmatrix} = \mu$.

ii) Assume that $(\mathcal{A},\mathcal{B})$ is controllable [38]. Then, $\max_{s\in\mathbf{C}} \mathrm{r}\begin{bmatrix} sI-\mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0\end{bmatrix} = \mu$ if and only if $\mathcal{C}=0$.

Lemma 3: Let E, $A \in \mathbf{R}^{n\times n}$, $\tilde{E}$, $\tilde{A} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$, $B \in \mathbf{R}^{m}$ and $\tilde{B} \in \mathbf{R}^{\tilde{m}}$. Assume that the pencil $sE-A$ and $s\tilde{E}-\tilde{A}$ are regular. Then, there are nonsingular matrices X, $Y \in \mathbf{R}^{n\times n}$ and $\tilde{X}$, $\tilde{Y} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$ such that

$$[X(sE-A)Y \quad | \quad XB] = \begin{bmatrix} sI-J & 0 & 0 & | & B_1 \\ 0 & sN_{22}-I & sN_{23} & | & B_2 \\ 0 & 0 & sN_{33}-I & | & 0 \end{bmatrix} \quad (9)$$

and

$$\left[\tilde{X}(s\tilde{E}-\tilde{A})\tilde{Y} \quad | \quad \tilde{X}\tilde{B}\right] = \begin{bmatrix} sI-\tilde{J} & 0 & | & \tilde{B}_1 \\ 0 & s\tilde{N}-I & | & \tilde{B}_2 \end{bmatrix} \quad (10)$$

where $J \in \mathbf{C}^{\tau_1\times\tau_1}$, $N_{22} \in \mathbf{C}^{\tau_2\times\tau_2}$, $N_{23} \in \mathbf{C}^{\tau_2\times\tau_3}$, $N_{33} \in \mathbf{C}^{\tau_3\times\tau_3}$, $B_1 \in \mathbf{C}^{\tau_1}$, $B_2 \in \mathbf{C}^{\tau_2}$, $\tilde{J} \in \mathbf{C}^{\tilde{\tau}_1\times\tilde{\tau}_1}$, $\tilde{N} \in \mathbf{C}^{\tilde{\tau}_2\times\tilde{\tau}_2}$, J is a matrix in the Jordan canonical form, the pair (N_{22}, B_2) is controllable, and $N = \begin{bmatrix} N_{22} & N_{23} \\ 0 & N_{33}\end{bmatrix}$ and $\tilde{N}$ are nilpotent.

Proof: The proof is given in Appendix.

A. Algebraic Characterization of Inclusion

Using nonsingular matrices $X, Y, \tilde{X}$, and $\tilde{Y}$ in Lemma 3, we have equivalent systems to $\mathbf{S}$ and $\tilde{\mathbf{S}}$

$$\mathbf{S}: \begin{cases} \dot{\xi}_1(t) = J\xi_1(t) + B_1u(t), \xi_1(0) = \xi_1^0 \\ N\begin{bmatrix}\dot{\xi}_2(t) \\ \dot{\xi}_3(t)\end{bmatrix} = \begin{bmatrix}\xi_2(t) \\ \xi_3(t)\end{bmatrix} + \begin{bmatrix} B_2 \\ 0\end{bmatrix}u(t) \\ \begin{bmatrix}\xi_2(0) \\ \xi_3(0)\end{bmatrix} = \begin{bmatrix}\xi_2^0 \\ \xi_3^0\end{bmatrix} \\ y(t) = CY\xi, \end{cases} \quad (11)$$

$$\tilde{\mathbf{S}}: \begin{cases} \dot{\tilde{\xi}}_1(t) = \tilde{J}\tilde{\xi}_1(t) + \tilde{B}_1\tilde{u}(t), \tilde{\xi}_1(0) = \tilde{\xi}_1^0 \\ \tilde{N}\dot{\tilde{\xi}}_2(t) = \tilde{\xi}_2(t) + \tilde{B}_2\tilde{u}(t), \tilde{\xi}_2(0) = \tilde{\xi}_2^0 \\ \tilde{y}(t) = \tilde{C}\tilde{Y}\tilde{\xi} \end{cases} \quad (12)$$

where

$$\xi = \begin{bmatrix}\xi_1 \\ \xi_2 \\ \xi_3\end{bmatrix} = Y^{-1}x, \quad \xi^0 = \begin{bmatrix}\xi_1^0 \\ \xi_2^0 \\ \xi_3^0\end{bmatrix} = Y^{-1}x^0 \quad (13)$$

$$\tilde{\xi} = \begin{bmatrix}\tilde{\xi}_1 \\ \tilde{\xi}_2\end{bmatrix} = \tilde{Y}^{-1}\tilde{x}, \quad \tilde{\xi}^0 = \begin{bmatrix}\tilde{\xi}_1^0 \\ \tilde{\xi}_2^0\end{bmatrix} = \tilde{Y}^{-1}\tilde{x}^0 \quad (14)$$

$\xi_1, \xi_1^0 \in \mathbf{R}^{\tau_1}$, $\xi_2, \xi_2^0 \in \mathbf{R}^{\tau_2}$, $\xi_3, \xi_3^0 \in \mathbf{R}^{\tau_3}$, $\tilde{\xi}_1, \tilde{\xi}_1^0 \in \mathbf{R}^{\tilde{\tau}_1}$, and $\tilde{\xi}_2, \tilde{\xi}_2^0 \in \mathbf{R}^{\tilde{\tau}_2}$.

The relations $\tilde{x}^0 = Vx^0$ and $x = V^+\tilde{x}$ in (5) and (6) become $\tilde{\xi}^0 = \tilde{Y}^{-1}VY\xi$ and $\xi = Y^{-1}V^+\tilde{Y}\tilde{\xi}$, and, hence, we need the following notation in the subsequent theorems:

$$CY = [C_1 \quad C_2 \quad C_3], \quad \tilde{C}\tilde{Y} = [\tilde{C}_1 \quad \hat{C}_2] \quad (15)$$

$$\tilde{Y}^{-1}VY = \begin{bmatrix}\mathcal{Z}_{11} & \mathcal{Z}_{12} & \mathcal{Z}_{13} \\ \mathcal{Z}_{21} & \mathcal{Z}_{22} & \mathcal{Z}_{23}\end{bmatrix} \quad (16)$$

$$Y^{-1}V^{(+)}\tilde{Y} = \left[Z^{(1)} \quad Z^{(2)}\right], \quad Z^{(i)} = \begin{bmatrix} Z_{1i} \\ Z_{2i} \\ Z_{3i}\end{bmatrix} \quad (17)$$

where $C_i \in \mathbf{C}^{l\times\tau_i}$, $i = 1,2,3$, $\tilde{C}_i \in \mathbf{C}^{\tilde{l}\times\tilde{\tau}_i}$, $i = 1,2$, and $Y^{-1}V^+\tilde{Y}\tilde{Y}^{-1}VY = I = [E^{(1)}\ E^{(2)}\ E^{(3)}]$

$$E^{(1)} = \begin{bmatrix} I \\ 0 \\ 0\end{bmatrix}, \quad E^{(2)} = \begin{bmatrix} 0 \\ I \\ 0\end{bmatrix}, \quad E^{(3)} = \begin{bmatrix} 0 \\ 0 \\ I\end{bmatrix}. \quad (18)$$

The solutions of (11) and (12) are given by

$$\begin{aligned} \xi(t;\xi^0,u) &= \left[e^{tJ}\xi_1^0 + \int_0^t e^{(t-s)J}B_1u(s)ds \right. \\ &\qquad \left. - \sum_{i=0}^{\tau_2+\tau_3-1} N^i\begin{bmatrix}B_2 \\ 0\end{bmatrix}u^{(i)}(t)\right] \\ &= \left[e^{tJ}\xi_1^0 + \int_0^t e^{(t-s)J}B_1u(s)ds \right. \\ &\qquad \left. - \sum_{i=0}^{\tau_2+\tau_3-1} \begin{bmatrix}N_{22}^iB_2 \\ 0\end{bmatrix}u^{(i)}(t)\right] \end{aligned} \quad (19)$$

and

$$\tilde{\xi}(t;\tilde{\xi}^0,\tilde{u}) = \left[e^{t\tilde{J}}\tilde{\xi}_1^0 + \int_0^t e^{(t-s)\tilde{J}}\tilde{B}_1\tilde{u}(s)ds \right. \left. - \sum_{i=0}^{\tilde{\tau}_2-1}\tilde{N}^i\tilde{B}_2\tilde{u}^{(i)}(t)\right] \quad (20)$$

respectively, where $u^{(i)}(t) = (d^i/dt^i)(u(t))$ and $\tilde{u}^{(i)}(t) = (d^i/dt^i)(\tilde{u}(t))$. Therefore, for any given inputs $u(\cdot)$ and $\tilde{u}(\cdot)$, the initial states ξ^0 and $\tilde{\xi}^0$ must satisfy

$$\begin{bmatrix}\xi_2^0 \\ \xi_3^0\end{bmatrix} = -\sum_{i=0}^{\tau_2+\tau_3-1}\begin{bmatrix}N_{22}^iB_2 \\ 0\end{bmatrix}u^{(i)}(0) \quad (21)$$

and

$$\tilde{\xi}_2^0 = -\sum_{i=0}^{\tilde{\tau}_2-1}\tilde{N}^i\tilde{B}_2\tilde{u}^{(i)}(0) \quad (22)$$

respectively, that is, ξ_2^0, ξ_3^0 and $\tilde{\xi}_2^0$ are determined by $u^{(i)}(0)$ and $\tilde{u}^{(i)}(0)$.

The necessary and sufficient conditions for (5) implying (6) are given by the following lemma:

Lemma 4: Let a triplet (V,L,T) be given. Then, the consistency conditions in (5) imply (6) for any fixed $u(t)$ and initial state x^0 if and only if the following conditions hold:

$$\mathcal{Z}_{21}=0, \quad \mathcal{Z}_{22}N_{22}^iB_2 = \tilde{N}^i\tilde{B}_2L, \quad i=0,1,\cdots \quad (23)$$

$$\begin{cases} Z^{(1)}\mathcal{Z}_{11} = E^{(1)}, \quad Z^{(2)}\mathcal{Z}_{22} = E^{(2)} \\ Z^{(1)}\tilde{J}^i\left[\tilde{J}\mathcal{Z}_{11} - \mathcal{Z}_{11}J) \quad \tilde{B}_1L - \mathcal{Z}_{11}B_1 \quad \mathcal{Z}_{12}\right] = 0 \\ \quad i=0,1,\cdots \end{cases} \quad (24)$$

and

$$\begin{cases} T^{(+)}\hat{C}_2\mathcal{Z}_{22} = C_2 \\ \left(T^{(+)}\tilde{C}_1 - C_1Z_{11}\right)\tilde{J}^i[\mathcal{Z}_{11} \quad \tilde{B}_1L - \mathcal{Z}_{11}B_1 \quad \mathcal{Z}_{12}] = 0 \\ \quad i=0,1,\cdots. \end{cases} \quad (25)$$

Proof: The corresponding solutions of (1) and (2) are given by $x(t;x^0,u) = Y\xi(t;\xi^0,u)$ and $\tilde{x}(t;\tilde{x}^0,\tilde{u}) = \tilde{Y}\tilde{\xi}(t;\tilde{\xi}^0,\tilde{u})$, where $\xi^0 = Y^{-1}x^0$, $\tilde{\xi}^0 = \tilde{Y}^{-1}\tilde{x}^0$.

Since $\tilde{\xi}^0 = \tilde{Y}^{-1}VY\xi^0$ and $\tilde{u} = Lu$, we obtain that the consistency conditions in (5) imply (6) for any fixed input $u(t)$ and initial state ξ_1^0 if and only if the following conditions hold for any $t \geq 0$:

1) $\tilde{\xi}^0 = \tilde{Y}^{-1}VY\xi^0$ is a consistent initial state of the system $\tilde{\mathbf{S}}$ with the input $\tilde{u}(t) = Lu(t)$.
From (21) and (22), we have $\tilde{\xi}^0 = \tilde{Y}^{-1}VY\xi^0$ for all ξ_1^0 and u if and only if

$$-\sum_{i=0}^{\tilde{\tau}_2-1}\tilde{N}^i\tilde{B}_2Lu^{(i)}(0) = \mathcal{Z}_{21}\xi_1^0 - \mathcal{Z}_{22}\sum_{i=0}^{\tau_2+\tau_3-1}N_{22}^iB_2u^{(i)}(0) \quad \forall\xi_1^0\in\mathbf{R}^{\tau_1},\ \forall u$$

which holds if and only if (23) holds.

2) $Y\xi(t;\xi^0,u) = V^+\tilde{Y}\tilde{\xi}(t;\tilde{\xi}_0,\tilde{u})$.
By (17)–(19), we have

$$\begin{aligned}
&E^{(1)}e^{tJ}\xi_1^0 + E^{(1)}\int_0^t e^{(t-s)J}B_1u(s)ds - E^{(2)}\sum_{i=0}^{\tau_2+\tau_3-1}N_{22}^iB_2u^{(i)}(0)\\
&= Z^{(1)}\left(e^{t\tilde{J}}\mathcal{Z}_{11}\xi_1^0 - \sum_{i=0}^{\tau_2+\tau_3-1}e^{t\tilde{J}}\mathcal{Z}_{12}N_{22}^iB_2u^{(i)}(0) + \int_0^t e^{(t-s)\tilde{J}}\tilde{B}_1Lu(s)ds\right)\\
&\quad - Z^{(2)}\sum_{i=0}^{\tilde{\tau}_2-1}\tilde{N}^i\tilde{B}_2Lu^{(i)}(t)
\end{aligned} \tag{26}$$

$$\Leftrightarrow \begin{cases}
E^{(1)}e^{tJ} = Z^{(1)}e^{t\tilde{J}}\mathcal{Z}_{11} & \forall t\geq 0\\
Z^{(1)}e^{t\tilde{J}}\mathcal{Z}_{12}N_{22}^iB_2 = 0 & \forall t\geq 0, i=0,1,\cdots\\
E^{(1)}e^{tJ}B_1 = Z^{(1)}e^{t\tilde{J}}\tilde{B}_1L & \forall t\geq 0\\
E^{(2)}N_{22}^iB_2 = Z^{(2)}\tilde{N}^i\tilde{B}_2L & i=0,1,\cdots,
\end{cases}$$

$$\Leftrightarrow \begin{cases}
E^{(1)}J^i = Z^{(1)}\tilde{J}^i\mathcal{Z}_{11} & i=0,1,\cdots\\
Z^{(1)}\tilde{J}^k\mathcal{Z}_{12}N_{22}^iB_2 = 0, & k,i=0,1,\cdots\\
E^{(1)}J^iB_1 = Z^{(1)}\tilde{J}^i\tilde{B}_1L & i=0,1,\cdots\\
E^{(2)}N_{22}^iB_2 = Z^{(2)}\tilde{N}^i\tilde{B}_2L & i=0,1,\cdots
\end{cases} \tag{27}$$

where we used facts that $\xi_1^0\in\mathbf{R}^{\tau_1}$ and $u(t)\in\mathcal{U}_{ad}$ can be arbitrary and that matrices N_{22} and $\tilde{N}$ are nilpotent. Since the pair (N_{22},B_2) is controllable, that is, $\mathrm{r}[B_2\ N_{22}B_2\ \cdots\ N_{22}^{\tau_2-1}B_2] = \tau_2$. Hence, due to (23), (27) holds if and only if (24) holds.

3) $CY\xi(t;\xi^0,u) = T^+\tilde{C}\tilde{Y}\tilde{\xi}(t;\tilde{\xi}_0,Lu)$.
Since $\tilde{\xi}^0 = \tilde{Y}^{-1}VY\xi^0$ and ξ^0 satisfies (21), we have $\tilde{\xi}_1^0 = \mathcal{Z}_{11}\xi_1^0 + \mathcal{Z}_{12}\sum_{i=0}^{\tau_2+\tau_3-1}N_{22}^iB_2u^{(i)}(0)$. Then, using (20) and notations $\tilde{C}\tilde{Y} = [\tilde{C}_1\ \hat{C}_2]$ and $Y^{-1}V^+\tilde{Y} = [Z^{(1)}\ Z^{(2)}]$, we have

$$\begin{aligned}
&CY\xi(t;\xi^0,u) = T^{(+)}\tilde{C}\tilde{Y}\tilde{\xi}(t;\tilde{\xi}_0,Lu)\\
\Leftrightarrow\ &\left(T^{(+)}\tilde{C}\tilde{Y} - CYY^{-1}V^{(+)}\tilde{Y}\right)\tilde{\xi}(t;\tilde{\xi}_0,\tilde{u}) = 0\\
\Leftrightarrow\ &\left(T^{(+)}\tilde{C}_1 - CYZ^{(1)}\right)\\
&\times\left(e^{t\tilde{J}}\mathcal{Z}_{11}\xi_1^0 - \sum_{i=0}^{\tau_2+\tau_3-1}e^{t\tilde{J}}\mathcal{Z}_{12}N_{22}^iB_2u^{(i)}(0) + \int_0^t e^{(t-s)\tilde{J}}\tilde{B}_1Lu(s)ds\right)\\
&- \left(T^{(+)}\hat{C}_2 - CYZ^{(2)}\right)\sum_{i=0}^{\tilde{\tau}_2-1}\tilde{N}^i\tilde{B}_2Lu^{(i)}(t) = 0\\
&\forall\xi_1^0\in\mathbf{R}^{\tau_1},\quad \forall u
\end{aligned}$$

$$\Leftrightarrow \begin{cases}
\left(T^{(+)}\tilde{C}_1 - CYZ^{(1)}\right)\tilde{J}^i\mathcal{Z}_{11} = 0\\
\left(T^{(+)}\tilde{C}_1 - CYZ^{(1)}\right)\tilde{J}^k\mathcal{Z}_{12}N_{22}^iB_2 = 0\\
\left(T^{(+)}\tilde{C}_1 - CYZ^{(1)}\right)\tilde{J}^i\tilde{B}_1L = 0\\
\left(T^{(+)}\hat{C}_2 - CYZ^{(2)}\right)\tilde{N}^i\tilde{B}_2L = 0\\
k,i = 0,1,\cdots.
\end{cases} \tag{28}$$

Because of both (23) and (24), (28) is equivalent to (25). Therefore, we conclude that for any fixed input $u(t)$ and initial state x^0, the consistency conditions in (5) imply (6) if and only if the conditions (23)–(25) are satisfied. This completes the proof of Lemma 4.□

The following lemma is a direct consequence of Lemmas 1 and 4, and is given without proof.

Lemma 5: Let a triplet (V,L,T) be given and let us use the notation in (9)–(17). Then, the consistency conditions in (5) imply (6) for any fixed $u(t)$ and initial state x^0 if and only if

$$\begin{cases}
Z^{(1)}\mathcal{Z}_{11} = E^{(1)},\quad Z^{(2)}\mathcal{Z}_{22} = E^{(2)},\quad \mathcal{Z}_{21} = 0\\
\max_{s\in\mathbf{C}}\mathrm{r}\left[\boldsymbol{Z}_J(s)\right] = \tilde{\tau}_1\\
\max_{s\in\mathbf{C}}\mathrm{r}\left[\boldsymbol{Z}_N(s)\right] = \tilde{\tau}_2+\tau_2\\
\max_{s\in\mathbf{C}}\mathrm{r}\left[\boldsymbol{Z}_T(s)\right] = \tilde{\tau}_1\\
T^{(+)}\hat{C}_2\mathcal{Z}_{22} = C_2
\end{cases} \tag{29}$$

where

$$\boldsymbol{Z}_J(s) = \begin{bmatrix} sI-\tilde{J} & \tilde{J}\mathcal{Z}_{11}-\mathcal{Z}_{11}J & \tilde{B}_1L-\mathcal{Z}_{11}B_1 & \mathcal{Z}_{12}\\ Z^{(1)} & 0 & 0 & 0\end{bmatrix}$$

$$\boldsymbol{Z}_N(s) = \begin{bmatrix} sI-\tilde{N} & 0 & \tilde{B}_2L\\ 0 & sI-N_{22} & B_2\\ I & -\mathcal{Z}_{22} & 0\end{bmatrix}$$

$$\boldsymbol{Z}_T(s) = \begin{bmatrix} sI-\tilde{J} & \mathcal{Z}_{11} & \tilde{B}_1L-\mathcal{Z}_{11}B_1 & \mathcal{Z}_{12}\\ T^{(+)}\tilde{C}_1-C_1Z_{11} & 0 & 0 & 0\end{bmatrix}.$$

The maximal rank of a matrix pencil over the complex field $\mathbf{C}$ can be determined by the generalized upper triangular form which can be computed by numerically stable algorithms [39]. Therefore, Lemma 5 provides a simple way to verify whether a given larger system $\tilde{\mathbf{S}}$ includes the given smaller system $\mathbf{S}$ under the transformations V, L and T.

In practical applications of the inclusion principle, it is required to construct a larger system $\tilde{\mathbf{S}}$ which includes the system $\mathbf{S}$. In the following theorem, we characterize and construct all such larger systems $\tilde{\mathbf{S}}$ explicitly.

Theorem 1: Let us use the notation in (9)–(15). Then, the system $\tilde{\mathbf{S}}$ includes the system $\mathbf{S}$ if and only if

$$\tilde{A} = \mathcal{X}\begin{bmatrix} J & 0 & 0 & \mathcal{A}_{14} \\ 0 & I & 0 & \mathcal{A}_{24} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} & \mathcal{A}_{34} \\ 0 & 0 & 0 & \mathcal{A}_{44} \end{bmatrix}\mathcal{Y} \tag{30}$$

$$\tilde{E} = \mathcal{X}\begin{bmatrix} I & 0 & 0 & \mathcal{E}_{14} \\ 0 & N_{22} & 0 & \mathcal{E}_{24} \\ 0 & \mathcal{E}_{32} & I & 0 \\ 0 & 0 & 0 & \mathcal{E}_{44} \end{bmatrix}\mathcal{Y} \tag{31}$$

$$\tilde{B} = \mathcal{X}\begin{bmatrix} B_1 & \mathcal{B}_{12} \\ B_2 & \mathcal{B}_{22} \\ \mathcal{B}_{31} & \mathcal{B}_{32} \\ 0 & \mathcal{B}_{42} \end{bmatrix}\begin{bmatrix} \hat{L} \\ P_L^\top \end{bmatrix} \tag{32}$$

$$\tilde{C} = [T \quad P_T]\begin{bmatrix} C_1 & C_2 & 0 & \mathcal{C}_{14} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} \end{bmatrix}\mathcal{Y} \tag{33}$$

where matrices $\mathcal{A}_{33} \in \mathbf{R}^{\tilde{\tau}_3\times\tilde{\tau}_3}$, $\mathcal{A}_{44}$, $\mathcal{E}_{44} \in \mathbf{R}^{\tilde{\tau}_4\times\tilde{\tau}_4}$, $\mathcal{B}_{31} \in \mathbf{R}^{\tilde{\tau}_3\times m}$, $\mathcal{C}_{14} \in \mathbf{R}^{l\times\tilde{\tau}_4}$, $[\mathcal{A}_{14}^T\ \mathcal{A}_{24}^\top\ \mathcal{A}_{34}^\top\ \mathcal{A}_{44}^\top]^\top$, $[\mathcal{E}_{14}^\top\ \mathcal{E}_{24}^\top\ \mathcal{E}_{44}^\top]^\top$, $[\mathcal{A}_{31}\ \mathcal{A}_{32}\ \mathcal{A}_{33}]$, $\mathcal{E}_{32}$, $[\mathcal{B}_{12}^\top\ \mathcal{B}_{22}^\top\ \mathcal{B}_{32}^\top\ \mathcal{B}_{42}^\top]^\top$, $\mathcal{B}_{31}, \mathcal{C}_{14}$ and $[\mathcal{C}_{21}\ \mathcal{C}_{22}\ \mathcal{C}_{23}\ \mathcal{C}_{24}]$ are arbitrary, $\tilde{\tau}_3$ and $\tilde{\tau}_4$ are arbitrary non-negative integers satisfying $\tau_1+\tau_2+\tilde{\tau}_3+\tilde{\tau}_4 = \tilde{n}$ and $\tau_3 \le \tilde{\tau}_4$, the pencil $s\mathcal{E}_{44} - \mathcal{A}_{44}$ is regular, $\mathcal{X}$ is any nonsingular matrix, $\mathcal{Y}$ is any nonsingular matrix of the form

$$\mathcal{Y} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \Upsilon_{33} & \Upsilon_{34} \\ 0 & 0 & \Upsilon_{43} & \Upsilon_{44} \end{bmatrix}\begin{bmatrix} Y^{-1}\hat{V} \\ P_V^\top \end{bmatrix} \tag{34}$$

with arbitrary matrices $\Upsilon_{33} \in \mathbf{R}^{\tilde{\tau}_3\times\tau_3}$, $\Upsilon_{44} \in \mathbf{R}^{\tilde{\tau}_4\times(\tilde{\tau}_3+\tilde{\tau}_4-\tau_3)}$ such that $\mathrm{r}(\Upsilon_{43}) = \tau_3$ and $\mathrm{r}(\Upsilon_{44}) = \tilde{\tau}_4 - \tau_3$, and $[\hat{V}^\top\ P_V]^\top$, $[\hat{L}^\top\ P_L]^\top$ and $[T\ P_T]$ are any nonsingular matrices satisfying $\hat{V}P_V = 0$, $\hat{L}P_L = 0$, $T^\top P_T = 0$, $P_V^\top P_V = I$, $P_L^\top P_L = I$ and $P_T^\top P_T = I$.

Proof: We prove the necessity first. Assume that the system $\tilde{\mathbf{S}}$ includes the system $\mathbf{S}$ and, thus, there exists a triplet (V, L, T) satisfying (3) and (4) such that the consistency conditions in (5) imply (6) for any fixed $u(t)$ and initial state x^0.

Let the QR factorizations of V, L and T be given by

$$\begin{cases} [\mathcal{P}_V \quad P_V]^\top V = \begin{bmatrix} \Sigma_V \\ 0 \end{bmatrix} \\ [\mathcal{P}_L \quad P_L]^\top L = \begin{bmatrix} \Sigma_L \\ 0 \end{bmatrix} \\ [\mathcal{P}_T \quad P_T]^\top T = \begin{bmatrix} \Sigma_T \\ 0 \end{bmatrix} \end{cases} \tag{35}$$

respectively, where $\mathcal{P}_V \in \mathbf{R}^{\tilde{n}\times n}$, $\mathcal{P}_L \in \mathbf{R}^{\tilde{m}\times m}$, $\mathcal{P}_T \in \mathbf{R}^{\tilde{l}\times l}$, $[\mathcal{P}_L\ P_L]$ and $[\mathcal{P}_T\ P_T]$ are orthogonal, matrices Σ_V, Σ_L and Σ_T are nonsingular. Obviously, matrices $\left[\begin{smallmatrix} V^+ \\ P_V^\top \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} L^+ \\ P_L^\top \end{smallmatrix}\right]$ and $[T\ P_T]$ are nonsingular, and $V^+P_V = 0$, $L^+P_L = 0$ and $T^\top P_T = 0$. Now, Lemma 5 implies that (29) holds. Obviously, $\mathcal{Z}_{11}$ and $\mathcal{Z}_{22}$ have full column rank. Let the QR factorizations of $\mathcal{Z}_{11}$ and $\mathcal{Z}_{22}$ be given by

$$[\mathcal{U} \quad U]^\top \mathcal{Z}_{11} = \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \begin{matrix} \}\tau_1 \\ \}\tilde{\tau}_1 - \tau_1 \end{matrix}$$

$$[\mathcal{Q} \quad Q]^\top \mathcal{Z}_{22} = \begin{bmatrix} R_2 \\ 0 \end{bmatrix} \begin{matrix} \}\tau_2 \\ \}\tilde{\tau}_2 - \tau_2 \end{matrix}$$

where R_1 and R_2 are nonsingular, $[\mathcal{U}\ U]$ and $[\mathcal{Q}\ Q]$ are orthogonal, $\mathcal{U} \in \mathbf{R}^{\tilde{\tau}_1\times\tau_1}$ and $\mathcal{Q} \in \mathbf{R}^{\tilde{\tau}_2\times\tau_2}$. Denote

$$[\mathcal{Z}_{11} \quad U]^{-1}\tilde{J}[\mathcal{Z}_{11} \quad U] = \overset{\begin{matrix}\tau_1 & \tilde{\tau}_1-\tau_1\end{matrix}}{\begin{bmatrix} \tilde{J}_{11} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \end{bmatrix}} \begin{matrix} \}\tau_1 \\ \}\tilde{\tau}_1 - \tau_1 \end{matrix}$$

$$[\mathcal{Z}_{22} \quad Q]^{-1}\tilde{N}[\mathcal{Z}_{22} \quad Q] = \overset{\begin{matrix}\tau_2 & \tilde{\tau}_2-\tau_2\end{matrix}}{\begin{bmatrix} \tilde{N}_{22} & \tilde{N}_{23} \\ \tilde{N}_{32} & \tilde{N}_{33} \end{bmatrix}} \begin{matrix} \}\tau_2 \\ \}\tilde{\tau}_2 - \tau_2 \end{matrix}$$

$$[\mathcal{Z}_{11} \quad U]^{-1}\tilde{B}_1 L = \begin{bmatrix} \tilde{\mathcal{B}}_1 \\ \tilde{\mathcal{B}}_2 \end{bmatrix} \begin{matrix} \}\tau_1 \\ \}\tilde{\tau}_1 - \tau_1 \end{matrix}$$

$$[\mathcal{Z}_{11} \quad U]^{-1}\mathcal{Z}_{12} = \begin{bmatrix} \tilde{\mathcal{Z}}_1 \\ \tilde{\mathcal{Z}}_2 \end{bmatrix} \begin{matrix} \}\tau_1 \\ \}\tilde{\tau}_1 - \tau_1 \end{matrix}.$$

We have from (29) that

$$\begin{aligned} \tilde{\tau}_1 &= \max_{s\in\mathbf{C}} \mathrm{r}\begin{bmatrix} sI - \tilde{J} & \tilde{J}\mathcal{Z}_{11} - \mathcal{Z}_{11}J & \tilde{B}_1 L - \mathcal{Z}_{11}B_1 & \mathcal{Z}_{12} \\ Z_{11} & 0 & 0 & 0 \\ Z_{21} & 0 & 0 & 0 \\ Z_{31} & 0 & 0 & 0 \end{bmatrix} \\ &= \max_{s\in\mathbf{C}} \mathrm{r}\begin{bmatrix} sI - \tilde{J}_{11} & -\tilde{J}_{12} & \tilde{J}_{11} - J & \tilde{\mathcal{B}}_1 - B_1 & \tilde{\mathcal{Z}}_1 \\ -\tilde{J}_{21} & sI - \tilde{J}_{22} & \tilde{J}_{21} & \tilde{\mathcal{B}}_2 & \tilde{\mathcal{Z}}_2 \\ I & Z_{11}U & 0 & 0 & 0 \\ 0 & Z_{21}U & 0 & 0 & 0 \\ 0 & Z_{31}U & 0 & & \end{bmatrix} \\ &= \max_{s\in\mathbf{C}} \mathrm{r}\left[\begin{matrix} -sZ_{11}U + JZ_{11}U - \tilde{J}_{12} & \tilde{J}_{11} - J & & \\ sI - \tilde{J}_{22} & \tilde{J}_{21} & & \\ Z_{21}U & 0 & 0 & 0 \\ Z_{31}U & 0 & & \end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix} \tilde{\mathcal{B}}_1 - B_1 & \tilde{\mathcal{Z}}_1 \\ \tilde{\mathcal{B}}_2 & \tilde{\mathcal{Z}}_2 \\ 0 & 0 \\ 0 & 0 \end{matrix}\right] + \tau_1 \\ &= \max_{s\in\mathbf{C}} \mathrm{r}\left[\begin{matrix} sI - \tilde{J}_{22} & \tilde{J}_{21} \\ JZ_{11}U - Z_{11}U\tilde{J}_{22} - \tilde{J}_{12} & \tilde{J}_{11} + Z_{11}U\tilde{J}_{21} - J \\ Z_{21}U & 0 \\ Z_{31}U & 0 \end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix} \tilde{\mathcal{B}}_2 & \tilde{\mathcal{Z}}_2 \\ \tilde{\mathcal{B}}_1 + Z_{11}U\tilde{\mathcal{B}}_2 - B_1 & \tilde{\mathcal{Z}}_1 + Z_{11}U\tilde{\mathcal{Z}}_2 \\ 0 & 0 \\ 0 & 0 \end{matrix}\right] + \tau_1 \end{aligned}$$

and

$$\begin{aligned} \tilde{\tau}_1 &= \max_{s\in\mathbf{C}} \mathrm{r}\begin{bmatrix} sI - \tilde{J} & \mathcal{Z}_{11} & \tilde{B}_1 L - \mathcal{Z}_{11}B_1 & \mathcal{Z}_{12} \\ T^{(+)}\tilde{C}_1 - C_1 Z_{11} & 0 & 0 & 0 \end{bmatrix} \\ &= \max_{s\in\mathbf{C}} \mathrm{r}\left[\begin{matrix} sI - \tilde{J}_{11} & -\tilde{J}_{21} \\ -\tilde{J}_{21} & sI - \tilde{J}_{22} \\ T^{(+)}\tilde{C}_1\mathcal{Z}_{11} - C_1 & T^{(+)}\tilde{C}_1 U - C_1 Z_{11} U \end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix} I & \tilde{\mathcal{B}}_1 - B_1 & \tilde{\mathcal{Z}}_1 \\ 0 & \tilde{\mathcal{B}}_2 & \tilde{\mathcal{Z}}_2 \\ 0 & 0 & 0 \end{matrix}\right] \\ &= \max_{s\in\mathbf{C}} \mathrm{r}\left[\begin{matrix} sI - \tilde{J}_{22} & \tilde{J}_{21} \\ T^{(+)}\tilde{C}_1 U - C_1 Z_{11} U & T^{(+)}\tilde{C}_1\mathcal{Z}_{11} - C_1 \end{matrix}\right. \\ &\qquad\qquad \left.\begin{matrix} \tilde{\mathcal{B}}_2 & \tilde{\mathcal{Z}}_2 \\ 0 & 0 \end{matrix}\right] + \tau_1 \end{aligned}$$

that is,

$$\max_{s\in\mathbf{C}} \mathrm{r}\begin{bmatrix} sI-\tilde{J}_{22} & \tilde{J}_{21} & \tilde{\mathcal{B}}_2 & \tilde{\mathcal{Z}}_2 \\ JZ_{11}U - Z_{11}U\tilde{J}_{22} - \tilde{J}_{12} & \tilde{J}_{11} + Z_{11}U\tilde{J}_{21} - J & \tilde{\mathcal{B}}_1 + Z_{11}U\tilde{\mathcal{B}}_2 - B_1 & \tilde{\mathcal{Z}}_1 + Z_{11}U\tilde{\mathcal{Z}}_2 \\ Z_{21}U & 0 & 0 & 0 \\ Z_{31}U & 0 & 0 & 0 \end{bmatrix} = \tilde{\tau}_1 - \tau_1,$$

and

$$\max_{s\in\mathbf{C}} \mathrm{r}\begin{bmatrix} sI-\tilde{J}_{22} & \tilde{J}_{21} & \tilde{\mathcal{B}}_2 & \tilde{\mathcal{Z}}_2 \\ T^{(+)}\tilde{C}_1U - C_1Z_{11}U & T^{(+)}\tilde{C}_1\mathcal{Z}_{11} - C_1 & 0 & 0 \end{bmatrix} = \tilde{\tau}_1 - \tau_1.$$

Thus, using (i) of Lemma 2 we obtain

$$\begin{cases} \tilde{J}_{11} = J - Z_{11}U\tilde{J}_{21}, \quad \tilde{\mathcal{B}}_1 = B_1 - Z_{11}U\tilde{\mathcal{B}}_2 \\ \tilde{\mathcal{Z}}_1 = -Z_{11}U\tilde{\mathcal{Z}}_2, \quad T^{(+)}\tilde{C}_1\mathcal{Z}_{11} = C_1, \end{cases} \tag{36}$$

$$\max_{s\in\mathbf{C}} \mathrm{r}\begin{bmatrix} sI-\tilde{J}_{22} & \tilde{J}_{21} & \tilde{\mathcal{B}}_2 & \tilde{\mathcal{Z}}_2 \\ JZ_{11}U - Z_{11}U\tilde{J}_{22} - \tilde{J}_{12} & 0 & 0 & 0 \\ Z_{21}U & 0 & 0 & 0 \\ Z_{31}U & 0 & 0 & 0 \\ T^{(+)}\tilde{C}_1U - C_1Z_{11}U & 0 & 0 & 0 \end{bmatrix} = \tilde{\tau}_1 - \tau_1. \tag{37}$$

Let orthogonal matrix $\mathcal{W}$ bring $(\tilde{J}_{22}, [\tilde{J}_{21}\ \tilde{\mathcal{B}}_2\ \tilde{\mathcal{Z}}_2])$ to its controllable stair-case form [39], i.e.,

$$\mathcal{W}^\top \tilde{J}_{22}\mathcal{W} = \begin{bmatrix} \overset{\tilde{\tau}_3}{\tilde{J}_{33}^{(0)}} & \overset{\hat{\tau}_4}{\hat{J}_{34}} \\ 0 & \hat{J}_{44} \end{bmatrix} \begin{matrix} \}\tilde{\tau}_3 \\ \}\hat{\tau}_4 \end{matrix} \tag{38}$$

$$\mathcal{W}^\top [\hat{J}_{21} \quad \hat{\mathcal{B}}_2 \quad \hat{\mathcal{Z}}_2] = \begin{bmatrix} \overset{\tau_1}{\mathcal{A}_{31}} & \overset{m}{\mathcal{B}_{31}} & \overset{\tau_2}{\Upsilon_{32}} \\ 0 & 0 & 0 \end{bmatrix} \tag{39}$$

where $(\tilde{J}_{33}^{(0)}, [\mathcal{A}_{31}\ \mathcal{B}_{31}\ \Upsilon_{32}])$ is controllable. Then (37) and (ii) of Lemma 2 yield

$$(JZ_{11}U - Z_{11}U\tilde{J}_{22} - \tilde{J}_{12})\mathcal{W} = [\overset{\tilde{\tau}_3}{0} \quad \overset{\hat{\tau}_4}{\hat{\mathcal{J}}_{14}}] \tag{40}$$

$$\left(T^{(+)}\tilde{C}_1U - C_1Z_{11}U\right)\mathcal{W} = [\overset{\tilde{\tau}_3}{0} \quad \overset{\hat{\tau}_4}{\hat{\mathcal{C}}_{14}}] \tag{41}$$

$$Z_{21}U\mathcal{W} = [\overset{\tilde{\tau}_3}{0} \quad \overset{\hat{\tau}_4}{\mathcal{X}_{24}}]$$

$$Z_{31}U\mathcal{W} = [\overset{\tilde{\tau}_3}{0} \quad \overset{\hat{\tau}_4}{\mathcal{X}_{34}}]. \tag{42}$$

Hence, using (36), (38), (39)–(42) and the property $Z^{(1)}\mathcal{Z}_{11} = E^{(1)}$, we obtain

$$\tilde{J} = \mathcal{Z}_U \begin{bmatrix} J - \mathcal{X}_{13}\mathcal{A}_{31} & J\mathcal{X}_{13} - \mathcal{X}_{13}\tilde{J}_{33}^{(0)} & \hat{J}_{14} - \mathcal{X}_{13}\hat{J}_{34} \\ \mathcal{A}_{31} & \tilde{J}_{33}^{(0)} & \hat{J}_{34} \\ 0 & 0 & \hat{J}_{44} \end{bmatrix} \times \mathcal{Z}_U^{-1}$$

$$= \mathcal{Z}_U \mathbb{X}_{13} \begin{bmatrix} J & 0 & \hat{J}_{14} \\ \mathcal{A}_{31} & \mathcal{A}_{33} & \hat{J}_{34} \\ 0 & 0 & \hat{J}_{44} \end{bmatrix} \mathbb{X}_{13}^{-1}\mathcal{Z}_U^{-1}$$

$$\mathcal{Z}_U = [\mathcal{Z}_{11} \quad U\mathcal{W}], \quad \mathbb{X}_{13} = \begin{bmatrix} I & -\mathcal{X}_{13} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$\tilde{B}_1 L = \mathcal{Z}_U \mathbb{X}_{13} \left[B_1^T \quad \mathcal{B}_{31}^T \quad 0\right]^T$$

$$\left(\text{i.e., } \tilde{B}_1 = \mathcal{Z}_U \mathbb{X}_{13} \begin{bmatrix} B_1 & \mathcal{B}_{12} \\ \mathcal{B}_{31} & \mathcal{B}_{32} \\ 0 & \tilde{B}_{42} \end{bmatrix} \begin{bmatrix} L^{(+)} \\ P_L^\top \end{bmatrix} \text{ for some } \mathcal{B}_{12}, \mathcal{B}_{32} \text{ and } \tilde{B}_{42}\right)$$

$$T^{(+)}\tilde{C}_1 = [C_1 \quad 0 \quad \hat{C}_{14}]\mathbb{X}_{13}^{-1}\mathcal{Z}_U^{-1}$$

$$\left(\text{i.e., } \tilde{C}_1 = [T \quad P_T] \begin{bmatrix} C_1 & 0 & \hat{C}_{14} \\ \mathcal{C}_{21} & \mathcal{C}_{23} & \hat{C}_{24} \end{bmatrix} \mathbb{X}_{13}^{-1}\mathcal{Z}_U^{-1} \text{ for some } \mathcal{C}_{21}, \mathcal{C}_{23} \text{ and } \hat{C}_{24}\right)$$

$$\mathcal{Z}_{12} = \mathcal{Z}_U \mathbb{X}_{13} \left[0 \quad \Upsilon_{32}^T \quad 0\right]^T$$

$$Z^{(1)} = \begin{bmatrix} I & 0 & \mathcal{X}_{14} \\ 0 & 0 & \mathcal{X}_{24} \\ 0 & 0 & \mathcal{X}_{34} \end{bmatrix} \mathbb{X}_{13}^{-1}\mathcal{Z}_U^{-1} \tag{43}$$

where

$$Z_{11}U\mathcal{W} = [\overset{\tilde{\tau}_3}{\mathcal{X}_{13}} \quad \overset{\hat{\tau}_4}{\mathcal{X}_{14}}],$$

and

$$\mathcal{A}_{33} = \tilde{J}_{33}^{(0)} - \mathcal{A}_{31}\mathcal{X}_{13}, \quad \hat{J}_{14} = J\mathcal{X}_{14} - \mathcal{X}_{14}\hat{J}_{44} - \hat{\mathcal{J}}_{14}$$
$$\hat{C}_{14} = C_1\mathcal{X}_{14} + \hat{\mathcal{C}}_{14}.$$

Similarly, using

$$\max_{s\in\mathbf{C}} \mathrm{r}\begin{bmatrix} sI - \tilde{N} & 0 & \tilde{B}_2 L \\ 0 & sI - N_{22} & B_2 \\ I & -\mathcal{Z}_{22} & 0 \end{bmatrix} = \tilde{\tau}_2 + \tau_2$$

$$T^{(+)}\hat{C}_2\mathcal{Z}_{22} = C_2, \quad Z^{(2)} = E^{(2)}, \mathcal{Z}_{21} = 0$$

we have that

$$[\mathcal{Z}_{22} \quad Q]^{-1}\tilde{N}[\mathcal{Z}_{22} \quad Q] = \begin{bmatrix} \overset{\tau_2}{N_{22}} & \overset{\hat{\tau}_5}{\tilde{N}_{25}} \\ 0 & \tilde{N}_{55} \end{bmatrix} \begin{matrix} \}\tau_2 \\ \}\hat{\tau}_5 \end{matrix}$$

$$[\mathcal{Z}_{22} \quad Q]^{-1}\tilde{B}_2[L \quad P_L] = \begin{bmatrix} B_2 & \mathcal{B}_{22} \\ 0 & \tilde{B}_{52} \end{bmatrix} \begin{matrix} \}\tau_2 \\ \}\hat{\tau}_5 \end{matrix}$$

$$\begin{bmatrix} T^{(+)} \\ P_T^\top \end{bmatrix} \hat{C}_2[\mathcal{Z}_{22} \quad Q] = \begin{bmatrix} \overset{\tau_2}{C_2} & \overset{\hat{\tau}_5}{\hat{C}_{15}} \\ \check{\mathcal{C}}_{22} & \hat{C}_{25} \end{bmatrix}$$

$$Z^{(2)}[\mathcal{Z}_{22} \quad Q] = \begin{bmatrix} 0 & \mathcal{X}_{15} \\ I & \mathcal{X}_{25} \\ 0 & \mathcal{X}_{35} \end{bmatrix}, \ \mathcal{Z}_{21} = 0. \tag{44}$$

Since $\tilde{N}$ is nilpotent, $\tilde{N}_{55}$ is nilpotent. Therefore, we obtain

$$\tilde{A} = \mathcal{X} \begin{bmatrix} J & 0 & 0 & \hat{J}_{14} & 0 \\ 0 & I & 0 & 0 & 0 \\ \mathcal{A}_{31} & 0 & \mathcal{A}_{33} & \hat{J}_{34} & 0 \\ 0 & 0 & 0 & \hat{J}_{44} & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \tilde{\mathcal{Y}} = \mathcal{X} \begin{bmatrix} J & 0 & 0 & \tilde{\mathcal{A}}_{14} \\ 0 & I & 0 & 0 \\ \mathcal{A}_{31} & 0 & \mathcal{A}_{33} & \tilde{\mathcal{A}}_{34} \\ 0 & 0 & 0 & \mathcal{A}_{44} \end{bmatrix} \tilde{\mathcal{Y}}$$

$$\tilde{\mathcal{A}}_{14} := [\hat{J}_{14} \quad 0]$$
$$\tilde{\mathcal{A}}_{34} := [\hat{J}_{34} \quad 0]$$
$$\mathcal{A}_{44} := \begin{bmatrix} \hat{J}_{44} & 0 \\ 0 & I \end{bmatrix}$$

$$\tilde{E} = \mathcal{X} \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & N_{22} & 0 & 0 & \tilde{N}_{25} \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & \tilde{N}_{55} \end{bmatrix} \tilde{\mathcal{Y}} = \mathcal{X} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & N_{22} & 0 & \tilde{\mathcal{E}}_{24} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & \mathcal{E}_{44} \end{bmatrix} \tilde{\mathcal{Y}}$$

$$\tilde{\mathcal{E}}_{24} := [0 \quad \tilde{N}_{25}]$$
$$\mathcal{E}_{44} := \begin{bmatrix} I & 0 \\ 0 & \tilde{N}_{55} \end{bmatrix}$$

$$\tilde{B} = \mathcal{X} \begin{bmatrix} B_1 & \mathcal{B}_{12} \\ B_2 & \mathcal{B}_{22} \\ \mathcal{B}_{31} & \mathcal{B}_{32} \\ 0 & \tilde{B}_{42} \\ 0 & \tilde{B}_{52} \end{bmatrix} \begin{bmatrix} L^{(+)} \\ P_L^{\top} \end{bmatrix} = \mathcal{X} \begin{bmatrix} B_1 & \mathcal{B}_{12} \\ B_2 & \mathcal{B}_{22} \\ \mathcal{B}_{31} & \mathcal{B}_{32} \\ 0 & \mathcal{B}_{42} \end{bmatrix} \begin{bmatrix} L^{(+)} \\ P_L^{\top} \end{bmatrix}, \quad \mathcal{B}_{42} := \begin{bmatrix} \tilde{B}_{42} \\ \tilde{B}_{52} \end{bmatrix}$$

$$\tilde{C} = [T \quad P_T] \begin{bmatrix} C_1 & C_2 & 0 & \hat{C}_{14} & \hat{C}_{15} \\ \mathcal{C}_{21} & \tilde{\mathcal{C}}_{22} & \mathcal{C}_{23} & \hat{C}_{24} & \hat{C}_{25} \end{bmatrix} \tilde{\mathcal{Y}} = [T \quad P_T] \begin{bmatrix} C_1 & C_2 & 0 & \tilde{\mathcal{C}}_{14} \\ \mathcal{C}_{21} & \tilde{\mathcal{C}}_{22} & \mathcal{C}_{23} & \tilde{\mathcal{C}}_{24} \end{bmatrix} \tilde{\mathcal{Y}}$$

where $\tilde{\tau}_4 := \hat{\tau}_4 + \hat{\tau}_5$,

$$\tilde{\mathcal{C}}_{14} := [\hat{C}_{14} \quad \hat{C}_{15}], \quad \tilde{\mathcal{C}}_{24} := [\hat{C}_{24} \quad \hat{C}_{25}]$$
$$\mathcal{X} = \tilde{X}^{-1} \mathcal{Z}_W \hat{\mathbb{X}}_{13}^{-1}, \quad \tilde{\mathcal{Y}} = \hat{\mathbb{X}}_{13} \mathcal{Z}_W^{-1} \tilde{Y}^{-1}$$
$$\mathcal{Z}_W = \begin{bmatrix} \mathcal{Z}_{11} & U\mathcal{W} & 0 & 0 \\ 0 & 0 & \mathcal{Z}_{22} & Q \end{bmatrix}$$
$$\hat{\mathbb{X}}_{13} = \begin{bmatrix} I & \mathcal{X}_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}^{-1}$$

$\mathcal{X}$ and $\tilde{\mathcal{Y}}$ are nonsingular. Note that

$$Y^{-1} V^{(+)} \tilde{Y} = \begin{bmatrix} I & 0 & 0 & \mathcal{X}_{14} & \mathcal{X}_{15} \\ 0 & I & 0 & \mathcal{X}_{24} & \mathcal{X}_{25} \\ 0 & 0 & 0 & \mathcal{X}_{34} & \mathcal{X}_{35} \end{bmatrix} \hat{\mathbb{X}}_{13} \mathcal{Z}_W^{-1}$$
$$\tilde{Y}^{-1} V Y = \mathcal{Z}_W \hat{\mathbb{X}}_{13}^{-1} \begin{bmatrix} I & 0 & \Upsilon_{13} \\ 0 & I & \Upsilon_{23} \\ 0 & \Upsilon_{32} & \hat{\Upsilon}_{33} \\ 0 & 0 & \hat{\Upsilon}_{43} \\ 0 & 0 & \hat{\Upsilon}_{53} \end{bmatrix}$$

where $[\Upsilon_{13}^{\top} \ \Upsilon_{23}^{\top} \ \Upsilon_{33}^{\top} \ \hat{\Upsilon}_{43}^{\top} \ \hat{\Upsilon}_{53}^{\top}] = [\mathcal{Z}_{13}^{\top} \ \mathcal{Z}_{23}^{\top}]$. Thus, $\tilde{\mathcal{Y}}$ satisfies

$$V^{(+)} = Y \begin{bmatrix} I & 0 & 0 & \Theta_{14} \\ 0 & I & 0 & \Theta_{24} \\ 0 & 0 & 0 & \Theta_{34} \end{bmatrix} \tilde{\mathcal{Y}}$$
$$\tilde{\mathcal{Y}} V Y = \begin{bmatrix} I & 0 & \Upsilon_{13} \\ 0 & I & \Upsilon_{23} \\ 0 & \Upsilon_{32} & \Upsilon_{33} \\ 0 & 0 & \Upsilon_{43} \end{bmatrix}$$
$$\begin{bmatrix} \Theta_{14} \\ \Theta_{24} \\ \Theta_{34} \end{bmatrix} := \begin{bmatrix} \mathcal{X}_{14} & \mathcal{X}_{15} \\ \mathcal{X}_{24} & \mathcal{X}_{25} \\ \mathcal{X}_{34} & \mathcal{X}_{35} \end{bmatrix}, \quad \Upsilon_{43} := \begin{bmatrix} \hat{\Upsilon}_{43} \\ \hat{\Upsilon}_{53} \end{bmatrix}.$$

Since, $V^{+} V = I$, $\tilde{\mathcal{Y}}$ must be of the form

$$\tilde{\mathcal{Y}} = \begin{bmatrix} I & 0 & -\Theta_{14}\Upsilon_{43} & -\Theta_{14}\Upsilon_{44} \\ 0 & I & -\Theta_{24}\Upsilon_{43} & -\Theta_{24}\Upsilon_{44} \\ 0 & \Upsilon_{32} & \Upsilon_{33} & \Upsilon_{34} \\ 0 & 0 & \Upsilon_{43} & \Upsilon_{44} \end{bmatrix} \begin{bmatrix} Y^{-1} V^{(+)} \\ P_V^{\top} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & -\Theta_{14} \\ 0 & I & 0 & -\Theta_{24} \\ 0 & \Upsilon_{32} & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \Upsilon_{33} & \Upsilon_{34} \\ 0 & 0 & \Upsilon_{43} & \Upsilon_{44} \end{bmatrix} \times \begin{bmatrix} Y^{-1} V^{(+)} \\ P_V^{\top} \end{bmatrix}$$

with $[\Upsilon_{34}^{\top} \ \Upsilon_{44}^{\top}]^{\top} \in \mathbf{R}^{(\tilde{\tau}_3 + \tilde{\tau}_4) \times (\tilde{\tau}_3 + \tilde{\tau}_4 - \tau_3)}$, and

$$\mathrm{r}(\Upsilon_{43}) = \tau_3, \quad \mathrm{r}(\Upsilon_{44}) = \tilde{\tau}_4 - \tau_3.$$

Hence, (30)–(33) and (34) hold with

$$\mathcal{A}_{14} = \tilde{\mathcal{A}}_{14} - J\Theta_{14}, \quad \mathcal{A}_{24} = -\Theta_{24} \quad \mathcal{A}_{32} = \mathcal{A}_{33}\Upsilon_{32}$$
$$\mathcal{A}_{34} = \tilde{\mathcal{A}}_{34} - \mathcal{A}_{31}\Theta_{14}$$
$$\mathcal{E}_{14} = -\Theta_{14}, \quad \mathcal{E}_{24} = \tilde{\mathcal{E}}_{24} - N_{22}\Theta_{24}, \quad \mathcal{E}_{32} = \Upsilon_{32}$$
$$\mathcal{C}_{14} = \tilde{\mathcal{C}}_{14} - C_1\Theta_{14} - C_2\Theta_{24}, \quad \mathcal{C}_{22} = \tilde{\mathcal{C}}_{22} + \mathcal{C}_{23}\Upsilon_{32}$$
$$\mathcal{C}_{24} = \tilde{\mathcal{C}}_{24} - \mathcal{C}_{21}\Theta_{14} - \tilde{\mathcal{C}}_{22}\Theta_{24}$$

and

$$\hat{V} = V^{(+)}, \quad \hat{L} = L^{(+)}.$$

Conversely, let the quadruplet $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{C})$ be given by (30)–(33) and (34), and let

$$V = \hat{V}^{(+)}, \quad L = \hat{L}^{(+)}.$$

Then, a direct verification gives that the consistency conditions in (5) imply (6) for any fixed $u(t)$ and initial state x^0. Hence, Definition 1 yields that the system $\tilde{\mathbf{S}}$ includes the system $\mathbf{S}$.□

An important issue in the expansion-contraction process has been the conditions under which structural properties of expansions and contractions, such as controllability, observability, and stabilizability remain invariant in the process. This issue has been raised in [22]–[24] regarding controllability and observability for standard linear time-invariant systems, and general conditions for their invariance have been formulated in [35], recently, it has been proved [22, Theorem 3.8] that the stability, controllability, stabilizability, observability, detectability, and the stability of the invariant zeros can be transmitted simultaneously from the original system to the expanded system under the inclusion principle. By a similar discussion, it can be shown that the same result is also true for descriptor systems.

We use Example 2 of [34] to illustrate Theorem 1. In this example, we consider a simple RLC electrical circuit.

Example 1: Let

$$E = \left[\begin{array}{c|cc|c} 1.1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$$A = \left[\begin{array}{c|cc|c} 0 & 1 & 0 & 0 \\ \hline 10^4 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 \end{array}\right]$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad C = [0 \quad 0 \quad 1 \quad 0].$$

According to the partitioning of E and A above, we take

$$V = \left[\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \quad \begin{array}{ll} L = T = 1 & n = 4 \\ m = l = 1 & \tilde{n} = 6 \\ \tilde{m} = \tilde{l} = 1. & \end{array}$$

By computing the form (9) we obtain

$$X = \begin{bmatrix} \frac{1}{1.1} & 0 & \frac{1}{1.1} & -\frac{1}{1.1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\tau_1 = 2, \quad \tau_2 = \tau_3 = 1$$

$$J = \begin{bmatrix} -\frac{2}{1.1} & -\frac{1}{1.1} \\ 10^4 & 0 \end{bmatrix}, \quad N_{22} = N_{33} = N_{23} = 0$$

$$B_1 = \begin{bmatrix} \frac{1}{1.1} \\ 0 \end{bmatrix}, \quad B_2 = -1$$

$$C_1 = [0 \quad 1], \quad C_2 = C_3 = 0.$$

Furthermore, the QR factorization of V in (35) provides

$$V^{(+)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_V = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Hence, by Theorem 1 and its proof we conclude that the system $\tilde{\mathbf{S}}$ includes the system $\mathbf{S}$ under the transformations V, L and T if and only if either

a)

$$\tilde{A} = \mathcal{X} \left[\begin{array}{cc|c|cc|c} -\frac{2}{1.1} & -\frac{1}{1.1} & 0 & 0 & 0 & a_{16} \\ 10^4 & 0 & 0 & 0 & 0 & a_{26} \\ \hline 0 & 0 & 1 & 0 & 0 & a_{36} \\ \hline a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ \hline 0 & 0 & 0 & 0 & 0 & a_{66} \end{array}\right] \mathcal{Y}$$

$$\tilde{E} = \mathcal{X} \left[\begin{array}{cc|c|cc|c} 1 & 0 & 0 & 0 & 0 & e_{16} \\ 0 & 1 & 0 & 0 & 0 & e_{26} \\ \hline 0 & 0 & 0 & 0 & 0 & e_{36} \\ \hline 0 & 0 & e_{43} & 1 & 0 & 0 \\ 0 & 0 & e_{53} & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & e_{66} \end{array}\right] \mathcal{Y}$$

$$\tilde{B} = \mathcal{X} \left[\begin{array}{c} \frac{1}{1.1} \\ 0 \\ \hline -1 \\ \hline b_{41} \\ b_{51} \\ \hline 0 \end{array}\right]$$

$$\tilde{C} = [0 \quad 1|0|0 \quad 0|c_{16}]\mathcal{Y}$$

$$\mathcal{Y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 2 & 0.5 & 0.5 & 0.5 & 0.5 & 0 \\ -2r_{44} & \frac{r_{45}}{\sqrt{2}} & \frac{r_{46}}{\sqrt{2}} & -\frac{r_{45}}{\sqrt{2}} & -\frac{r_{46}}{\sqrt{2}} & r_{44} \\ -2r_{54} & \frac{r_{55}}{\sqrt{2}} & \frac{r_{56}}{\sqrt{2}} & -\frac{r_{55}}{\sqrt{2}} & -\frac{r_{56}}{\sqrt{2}} & r_{54} \\ -2r_{64} & 0 & 0 & 0 & 0 & r_{64} \end{bmatrix}$$

$$|a_{66}| + |e_{66}| \neq 0, r_{64} \neq 0, \det \begin{bmatrix} r_{45} & r_{46} \\ r_{55} & r_{56} \end{bmatrix} \neq 0 \quad (45)$$

or

b)

$$\tilde{A} = \mathcal{X} \left[\begin{array}{cc|c|c|cc} -\frac{2}{1.1} & -\frac{1}{1.1} & 0 & 0 & a_{15} & a_{16} \\ 10^4 & 0 & 0 & 0 & a_{25} & a_{26} \\ \hline 0 & 0 & 1 & 0 & a_{35} & a_{36} \\ \hline a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ \hline 0 & 0 & 0 & 0 & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{array}\right] \mathcal{Y}$$

$$\tilde{E} = \mathcal{X} \left[\begin{array}{cc|c|c|cc} 1 & 0 & 0 & 0 & e_{15} & e_{16} \\ 0 & 1 & 0 & 0 & e_{25} & e_{26} \\ \hline 0 & 0 & 0 & 0 & e_{35} & e_{36} \\ \hline 0 & 0 & e_{43} & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & e_{55} & e_{56} \\ 0 & 0 & 0 & 0 & e_{65} & e_{66} \end{array}\right] \mathcal{Y}$$

$$\tilde{B} = \mathcal{X}\begin{bmatrix} \frac{1}{1.1} \\ \hline 0 \\ \hline -1 \\ \hline b_{41} \\ \hline 0 \\ 0 \end{bmatrix}$$

$$\tilde{C} = [0 \quad 1|0|0|c_{15} \quad c_{16}]\mathcal{Y}$$

$$s\begin{bmatrix} e_{55} & e_{56} \\ e_{65} & e_{66} \end{bmatrix} - \begin{bmatrix} a_{55} & a_{56} \\ a_{65} & a_{66} \end{bmatrix} \text{ is regular}$$

$$\mathcal{Y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 2 & 0.5 & 0.5 & 0.5 & 0.5 & 0 \\ -2r_{44} & \frac{r_{45}}{\sqrt{2}} & \frac{r_{46}}{\sqrt{2}} & -\frac{r_{45}}{\sqrt{2}} & -\frac{r_{46}}{\sqrt{2}} & r_{44} \\ -2r_{54} & \frac{r_{55}}{\sqrt{2}} & \frac{r_{56}}{\sqrt{2}} & -\frac{r_{55}}{\sqrt{2}} & -\frac{r_{56}}{\sqrt{2}} & r_{54} \\ -2r_{64} & \frac{r_{65}}{\sqrt{2}} & \frac{r_{66}}{\sqrt{2}} & -\frac{r_{65}}{\sqrt{2}} & -\frac{r_{66}}{\sqrt{2}} & r_{64} \end{bmatrix} \quad (46)$$

$$\mathrm{r}\begin{bmatrix} r_{54} \\ r_{64} \end{bmatrix} = \mathrm{r}\begin{bmatrix} r_{55} & r_{56} \\ r_{65} & r_{66} \end{bmatrix} = 1 \quad (47)$$

$$\begin{bmatrix} r_{44} & r_{45} & r_{46} \\ r_{54} & r_{55} & r_{56} \\ r_{64} & r_{65} & r_{66} \end{bmatrix} \text{ is nonsingular} \quad (48)$$

or

c)

$$\tilde{A} = \mathcal{X}\left[\begin{array}{cc|c|ccc} -\frac{2}{1.1} & -\frac{1}{1.1} & 0 & a_{14} & a_{15} & a_{16} \\ 10^4 & 0 & 0 & a_{24} & a_{25} & a_{26} \\ \hline 0 & 0 & 1 & a_{34} & a_{35} & a_{36} \\ \hline 0 & 0 & 0 & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & a_{64} & a_{65} & a_{66} \end{array}\right]\mathcal{Y}$$

$$\tilde{E} = \mathcal{X}\left[\begin{array}{cc|c|ccc} 1 & 0 & 0 & e_{14} & e_{15} & e_{16} \\ 0 & 1 & 0 & e_{24} & e_{25} & e_{26} \\ \hline 0 & 0 & 0 & e_{34} & e_{35} & e_{36} \\ \hline 0 & 0 & 0 & e_{44} & e_{45} & e_{46} \\ 0 & 0 & 0 & e_{54} & e_{55} & e_{56} \\ 0 & 0 & 0 & e_{64} & e_{65} & e_{66} \end{array}\right]\mathcal{Y}$$

$$\tilde{B} = \mathcal{X}\begin{bmatrix} \frac{1}{1.1} \\ 0 \\ \hline -1 \\ \hline 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{C} = [0 \quad 1|0|c_{14} \quad c_{15} \quad c_{16}]\mathcal{Y}$$

$$s\begin{bmatrix} e_{44} & e_{45} & e_{46} \\ e_{54} & e_{55} & e_{56} \\ e_{64} & e_{65} & e_{66} \end{bmatrix} - \begin{bmatrix} a_{44} & a_{45} & a_{46} \\ a_{54} & a_{55} & a_{56} \\ a_{64} & a_{65} & a_{66} \end{bmatrix} \text{ is regular}$$

$\mathcal{Y}$ is given by (46) and (48).

Note that

- For $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{C})$ given in (a), if $e_{66} \neq 0$ or $e_{66} = 0$, respectively, then $\tilde{\mathbf{S}}$ has 0 or 1 uncontrollable pole at infinity, respectively;
- For $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{C})$ given in (b), if the pencil $s\begin{bmatrix} e_{55} & e_{56} \\ e_{65} & e_{66} \end{bmatrix} - \begin{bmatrix} a_{55} & a_{56} \\ a_{65} & a_{66} \end{bmatrix}$ has 0, 1 or 2 infinite eigenvalues, respectively, the system $\tilde{\mathbf{S}}$ has 0, 1 or 2 uncontrollable poles at infinity;
- For $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{C})$ given in (c), and if the pencil $s\begin{bmatrix} e_{44} & e_{45} & e_{46} \\ e_{54} & e_{55} & e_{56} \\ e_{64} & e_{65} & e_{66} \end{bmatrix} - \begin{bmatrix} a_{44} & a_{45} & a_{46} \\ a_{54} & a_{55} & a_{56} \\ a_{64} & a_{65} & a_{66} \end{bmatrix}$ has 0, 1, 2 or 3 infinite eigenvalues, respectively, then the system $\tilde{\mathbf{S}}$ has 0, 1, 2 or 3 uncontrollable poles at infinity, respectively.

However, the system $\mathbf{S}$ has only one uncontrollable pole at infinity, thus, the controllability of the system $\mathbf{S}$ at infinity plays an important role in the inclusion.

A wide variety of applications of the expansion-contraction concept relies on decentralized control with overlapping information structure constraints. When a plant is composed of interconnected subsystems that share common parts, decentralized control laws, which utilize the state variables of the overlapping parts, are superior to disjoint decentralized control laws. Hence, an interesting idea is to explore the freedom in the expansion process in order to make the interconnection (off-diagonal) block matrices of the expanded system $\tilde{\mathbf{S}}$ as sparse as possible [16], [17], thus enhancing decentralized control strategies for design of the overall system. For Example 1, this can be done by choosing the arbitrary entries in matrices $\tilde{A}$, $\tilde{E}$, $\mathcal{Y}$ and $\mathcal{X}$ such that $\|\tilde{A}(4:6, 1:3)\|_F + \|\tilde{E}(4:6, 1:3)\|_F$ achieves the minimum. By solving this optimization problem with $\tilde{A}$ and $\tilde{E}$ being of the forms in the case (c) above, we found that we can choose

$$\mathcal{X} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 1 & 0 \\ 2 & 0 & \frac{2}{1.1} & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{Y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 2 & 0.5 & 0.5 & 0.5 & 0.5 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which lead to

$$\tilde{E} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\tilde{A} = \left[\begin{array}{ccc|ccc} -\frac{2}{1.1} & 0 & -\frac{1}{2.2} & 0 & -\frac{1}{2.2} & 0 \\ 10^4 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0.5 & 0.5 & 0.5 & 0.5 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -\frac{10^4}{\sqrt{2}} \\ 0 & 0 & 0 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 0 & 0 & 0 & \frac{2}{1.1} & 0 & 0 \end{array}\right].$$

Clearly, the resulting expanded system $\tilde{\mathbf{S}}$ has two 3 dimensional disjoint subsystems!

B. Aggregation and Restriction

Two special cases of the contraction, namely *aggregation* and *restriction* [1]–[3], have been studied for linear time-invariant systems extensively in the existing literature. These two concepts can be extended to descriptor systems as follows.

Definition 3:

a) The system $\mathbf{S}$ is an aggregation of the system $\tilde{\mathbf{S}}$ if there exists a triplet (V, L, T) satisfying (3) and (4) such that for any fixed input $\tilde{u}(t)$ and initial state $\tilde{\xi}_0$, the consistency conditions

$$x^0 = V^{(+)}\tilde{\xi}_0, \quad u(t) = L^{(+)}\tilde{u}(t), \quad \forall t \geq 0 \tag{49}$$

imply

$$\begin{aligned} x(t; x^0, u) &= V^{(+)}\tilde{\xi}(t; \tilde{\xi}_0, \tilde{u}) \\ y\left[x(t; x^0, u)\right] &= T^{(+)}\tilde{y}\left[\tilde{\xi}(t; \tilde{\xi}_0, \tilde{u})\right], \quad \forall t \geq 0. \end{aligned} \tag{50}$$

b) The system $\mathbf{S}$ is a restriction of the system $\tilde{\mathbf{S}}$ if there exists a triplet (V, L, T) satisfying (3) and (4) such that for any fixed input $u(t)$ and initial state x^0, the consistency conditions

$$\tilde{\xi}_0 = Vx^0, \quad \tilde{u}(t) = Lu(t), \forall t \geq 0,$$

imply

$$\begin{aligned} \tilde{\xi}(t; \tilde{\xi}_0, \tilde{u}) &= Vx(t; x^0, u) \\ \tilde{y}\left[\tilde{\xi}(t; \tilde{\xi}_0, \tilde{u})\right] &= Ty\left[x(t; x^0, u)\right] \quad \forall t \geq 0. \end{aligned} \tag{51}$$

Aggregation and restriction for descriptor systems are characterized explicitly by the following result.

Theorem 2: Let us use the notation in (9)–(15). Then,

a) The system $\mathbf{S}$ is an aggregation of the system $\tilde{\mathbf{S}}$ if and only if there is a coordinate frame where the quadruplet $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{C})$ has the matrices

$$\begin{aligned} \tilde{A} &= \mathcal{X}\begin{bmatrix} J & 0 & 0 & 0 & \mathcal{A}_{15} \\ 0 & I & 0 & 0 & \mathcal{A}_{25} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} & \tilde{\mathcal{A}}_{34} & \mathcal{A}_{35} \\ 0 & 0 & 0 & \mathcal{A}_{44} & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}\mathcal{Y} \\ \tilde{E} &= \mathcal{X}\begin{bmatrix} I & 0 & 0 & 0 & \mathcal{E}_{15} \\ 0 & N_{22} & 0 & 0 & \mathcal{E}_{25} \\ 0 & \mathcal{E}_{32} & I & 0 & 0 \\ 0 & 0 & 0 & \mathcal{E}_{44} & \mathcal{E}_{45} \\ 0 & 0 & 0 & 0 & \mathcal{E}_{55} \end{bmatrix}\mathcal{Y} \\ \tilde{B} &= \mathcal{X}\begin{bmatrix} B_1 & 0 \\ B_2 & 0 \\ \mathcal{B}_{31} & \mathcal{B}_{32} \\ 0 & \mathcal{B}_{42} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{L} \\ P_L^\top \end{bmatrix} \\ \tilde{C} &= [T \quad P_T]\begin{bmatrix} C_1 & C_2 & 0 & 0 & \mathcal{C}_{15} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} & \mathcal{C}_{25} \end{bmatrix}\mathcal{Y} \end{aligned} \tag{52}$$

such that $\mathcal{A}_{33} \in \mathbf{R}^{\tilde{\tau}_3\times\tilde{\tau}_3}$, $\mathcal{A}_{44}, \mathcal{E}_{44} \in \mathbf{R}^{\tilde{\tau}_4\times\tilde{\tau}_4}$, $\mathcal{E}_{55} \in \mathbf{R}^{\tilde{\tau}_5\times\tilde{\tau}_5}$, $\mathcal{B}_{31} \in \mathbf{R}^{\tilde{\tau}_3\times m}$, $\mathcal{B}_{42} \in \mathbf{R}^{\tilde{\tau}_4\times(\tilde{m}-m)}$, $\mathcal{C}_{15} \in \mathbf{R}^{l\times\tilde{\tau}_5}$, and matrices $[\mathcal{A}_{15}^\top\ \mathcal{A}_{25}^\top\ \mathcal{A}_{35}^\top]$, $[\mathcal{A}_{31}\ \mathcal{A}_{32}\ \mathcal{A}_{33}\ \mathcal{A}_{34}]$, $\mathcal{A}_{44}$, $[\mathcal{E}_{15}^\top\ \mathcal{E}_{25}^\top]$, $\mathcal{E}_{32}$, $[\mathcal{E}_{44}\ \mathcal{E}_{45}]$, $\mathcal{E}_{55}$, $\mathcal{B}_{31}$, $\left[\begin{smallmatrix}\mathcal{B}_{32}\\ \mathcal{B}_{42}\end{smallmatrix}\right]$, $\mathcal{C}_{15}$ and $[\mathcal{C}_{21}\ \mathcal{C}_{22}\ \mathcal{C}_{23}\ \mathcal{C}_{24}\ \mathcal{C}_{25}]$ are arbitrary, $\tilde{\tau}_3$, $\tilde{\tau}_4$ and $\tilde{\tau}_5$ are any arbitrary non-negative integers satisfying $\tau_1 + \tau_2 + \tilde{\tau}_3 + \tilde{\tau}_4 + \tilde{\tau}_5 = \tilde{n}$, $\tau_3 \leq \tilde{\tau}_5$, the pencil $s\mathcal{E}_{44} - \mathcal{A}_{44}$ is regular, $\mathcal{E}_{55}$ is nilpotent, $\mathcal{X}$ is any nonsingular matrix, $\mathcal{Y}$ is any nonsingular matrix of the form

$$\mathcal{Y} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & \Upsilon_{33} & \Upsilon_{34} & \Upsilon_{35} \\ 0 & 0 & \Upsilon_{43} & \Upsilon_{44} & \Upsilon_{45} \\ 0 & 0 & \Upsilon_{53} & \Upsilon_{54} & \Upsilon_{55} \end{bmatrix}\begin{bmatrix} Y^{-1}\hat{V} \\ P_V^\top \end{bmatrix} \tag{53}$$

with any arbitrary $[\Upsilon_{33}^\top\ \Upsilon_{43}^\top\ \Upsilon_{53}^\top]^\top \in \mathbf{R}^{\sum_{i=3}^5 \tilde{\tau}_i\times\tau_3}$, $\begin{bmatrix} \Upsilon_{34} & \Upsilon_{35} \\ \Upsilon_{44} & \Upsilon_{45} \\ \Upsilon_{54} & \Upsilon_{55} \end{bmatrix} \in \mathbf{R}^{\sum_{i=3}^5 \tilde{\tau}_i\times(\sum_{i=3}^5 \tilde{\tau}_i-\tau_3)}$, $\mathrm{r}(\Upsilon_{53}) = \tau_3$, and $\mathrm{r}[\Upsilon_{54}\ \Upsilon_{55}] = \tilde{\tau}_5 - \tau_3$, and matrices $[\hat{V}^\top\ P_V]^\top$, $[\hat{L}^\top\ P_L]^\top$ and $[T\ P_T]$ are any nonsingular matrices with $\hat{V}P_V = 0$, $\hat{L}P_L = 0$, $T^\top P_T = 0$, $P_V^\top P_V = I$, $P_L^\top P_L = I$, and $P_T^\top P_T = I$.

b) The system $\mathbf{S}$ is a restriction of the system $\tilde{\mathbf{S}}$ if and only if there is a coordinate frame where the quadruplet $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{C})$ has the matrices

$$\begin{aligned} \tilde{A} &= \mathcal{X}\begin{bmatrix} J & 0 & \mathcal{A}_{13} \\ 0 & I & \mathcal{A}_{23} \\ 0 & 0 & \mathcal{A}_{33} \end{bmatrix}\begin{bmatrix} Y^{-1}\hat{V} \\ P_V^\top \end{bmatrix} \\ \tilde{E} &= \mathcal{X}\begin{bmatrix} I & 0 & \mathcal{E}_{13} \\ 0 & N_{22} & \mathcal{E}_{23} \\ 0 & 0 & \mathcal{E}_{33} \end{bmatrix}\begin{bmatrix} Y^{-1}\hat{V} \\ P_V^\top \end{bmatrix} \\ \tilde{B} &= \mathcal{X}\begin{bmatrix} B_1 & \mathcal{B}_{12} \\ B_2 & \mathcal{B}_{22} \\ 0 & \mathcal{B}_{32} \end{bmatrix}\begin{bmatrix} \hat{L} \\ P_L^\top \end{bmatrix} \\ \tilde{C} &= [T \quad P_T]\begin{bmatrix} C_1 & C_2 & \mathcal{C}_{13} \\ 0 & 0 & \mathcal{C}_{23} \end{bmatrix}\begin{bmatrix} Y^{-1}\hat{V} \\ P_V^\top \end{bmatrix} \end{aligned}$$

such that matrices $\mathcal{A}_{13}, \mathcal{E}_{13} \in \mathbf{R}^{\tau_1\times\tilde{\tau}_3}$, $\mathcal{A}_{23}, \mathcal{E}_{23} \in \mathbf{R}^{\tau_2\times\tilde{\tau}_3}$, $\mathcal{A}_{33}, \mathcal{E}_{33} \in \mathbf{R}^{\tilde{\tau}_3\times\tilde{\tau}_3}$, $[\mathcal{B}_{12}^\top\ \mathcal{B}_{22}^\top\ \mathcal{B}_{32}^\top]^\top \in \mathbf{R}^{\tilde{n}\times(\tilde{m}-m)}$, $[\mathcal{C}_{13}^\top\ \mathcal{C}_{23}^\top]^\top \in \mathbf{R}^{\tilde{l}\times\tilde{\tau}_3}$ are arbitrary, $\tilde{\tau}_3 = \tilde{n} - \tau_1 - \tau_2$, the pencil $s\mathcal{E}_{33} - \mathcal{A}_{33}$ is regular, $\mathcal{X}$ is any nonsingular matrix, and matrices $[\hat{V}^\top\ P_V]^\top$, $[\hat{L}^\top\ P_L]^\top$ and $[T\ P_T]$ are any nonsingular matrices with $\hat{V}P_V = 0$, $\hat{L}P_L = 0$, $T^\top P_T = 0$, $P_V^\top P_V = I$, $P_L^\top P_L = I$, and $P_T^\top P_T = I$.

Proof: The proof is given in Appendix.

C. Explicit Algebraic Characterizations for Contractibility

The problem of contractibility of control laws [2], [11], [18] is important because, in applications of overlapping decentralized control to complex systems, the feedback control design is carried out in the expanded space. Then, the obtained expanded control laws are contracted to the smaller space for implementation in the original system. The needed result is the following:

Lemma 6: Let a triplet (V, L, T) be given and let us use the notation in (9)–(17). Denote

$$KY = [\overset{\tau_1}{K_1}\quad \overset{\tau_2}{K_2}\quad \overset{\tau_3}{K_2}], \quad \tilde{K}\tilde{Y} = [\overset{\tilde{\tau}_1}{\hat{K}_1}\quad \overset{\tilde{\tau}_2}{\hat{K}_2}]. \tag{54}$$

Then, the control law

$$\tilde{u} = -\tilde{K}\tilde{x} + \tilde{v}$$

for system $\tilde{\mathbf{S}}$ is contractible to the control law

$$u = -Kx + v$$

for implementation in system $\mathbf{S}$ under transformations V, L and T, if and only if either

$$\begin{aligned}
&Z^{(1)}\mathcal{Z}_{11} = E^{(1)}, \quad Z^{(2)}\mathcal{Z}_{22} = E^{(2)}, \quad \mathcal{Z}_{21} = 0\\
&\max_{s\in\mathbf{C}} \mathrm{r}\begin{bmatrix} sI-\tilde{J} & \tilde{J}\mathcal{Z}_{11}-\mathcal{Z}_{11}J & \tilde{B}_1L-\mathcal{Z}_{11}B_1 & \mathcal{Z}_{12}\\ Z_{11} & 0 & 0 & 0\\ Z_{21} & 0 & 0 & 0\\ Z_{31} & 0 & 0 & 0\end{bmatrix} = \tilde{\tau}_1\\
&\max_{s\in\mathbf{C}} \mathrm{r}\begin{bmatrix} sI-\tilde{N} & 0 & \tilde{B}_2L\\ 0 & sI-N_{22} & B_2\\ I & -\mathcal{Z}_{22} & 0\end{bmatrix} = \tilde{\tau}_2+\tau_2\\
&\max_{s\in\mathbf{C}} \mathrm{r}\begin{bmatrix} sI-\tilde{J} & \mathcal{Z}_{11} & \tilde{B}_1L-\mathcal{Z}_{11}B_1 & \mathcal{Z}_{12}\\ \hat{K}_1-LK_1Z_{11} & 0 & 0 & 0\end{bmatrix} = \tilde{\tau}_1\\
&\hat{K}_2\mathcal{Z}_{22} = LK_2;
\end{aligned} \tag{55}$$

or

$$\begin{aligned}
&Z^{(1)}\mathcal{Z}_{11} = E^{(1)}, \quad Z^{(2)}\mathcal{Z}_{22} = E^{(2)}, \quad \mathcal{Z}_{21} = 0\\
&\max_{s\in\mathbf{C}} \mathrm{r}\begin{bmatrix} sI-\tilde{J} & \tilde{J}\mathcal{Z}_{11}-\mathcal{Z}_{11}J & \tilde{B}_1-\mathcal{Z}_{11}B_1L^{(+)} & \mathcal{Z}_{12}\\ Z_{11} & 0 & 0 & 0\\ Z_{21} & 0 & 0 & 0\\ Z_{31} & 0 & 0 & 0\end{bmatrix} = \tilde{\tau}_1\\
&\max_{s\in\mathbf{C}} \mathrm{r}\begin{bmatrix} sI-\tilde{N} & 0 & \tilde{B}_2\\ 0 & sI-N_{22} & B_2L^{(+)}\\ I & -\mathcal{Z}_{22} & 0\end{bmatrix} = \tilde{\tau}_2+\tau_2\\
&\max_{s\in\mathbf{C}} \mathrm{r}\begin{bmatrix} sI-\tilde{J} & \mathcal{Z}_{11} & \tilde{B}_1-\mathcal{Z}_{11}B_1L^{(+)} & \mathcal{Z}_{12}\\ L^{(+)}\hat{K}_1-K_1Z_{11} & 0 & 0 & 0\end{bmatrix} = \tilde{\tau}_1\\
&L^{(+)}\hat{K}_2\mathcal{Z}_{22} = K_2.
\end{aligned} \tag{56}$$

Proof: The proof is given in Appendix.

We are now ready to characterize the contractibility explicitly in the following theorem.

Theorem 3: Let us use the notation in (9)–(15). Then, the control law

$$\tilde{u} = -\tilde{K}\tilde{x} + \tilde{v}$$

for system $\tilde{\mathbf{S}}$ is contractible to the control law

$$u = -Kx + v$$

for implementation in system $\mathbf{S}$, if and only if either the triplet $(\tilde{E}, \tilde{A}, \tilde{B})$ is given by (30)–(33) and (34), and

$$\tilde{K} = \begin{bmatrix}\hat{L}^{(+)} & P_L\end{bmatrix}\begin{bmatrix} K_1 & K_2 & 0 & \mathcal{K}_{14}\\ 0 & 0 & 0 & \mathcal{K}_{24}\end{bmatrix}\mathcal{Y} \tag{57}$$

with any arbitrary $[\mathcal{K}_{14}^\top\ \mathcal{K}_{24}^\top]^\top \in \mathbf{R}^{\tilde{m}\times\tilde{\tau}_4}$, or there is a coordinate frame where the quadruplet $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{K})$ has the matrices

$$\begin{aligned}
\tilde{A} &= \mathcal{X}\begin{bmatrix} J & 0 & 0 & 0 & \mathcal{A}_{15}\\ 0 & I & 0 & 0 & \mathcal{A}_{25}\\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} & \mathcal{A}_{34} & \mathcal{A}_{35}\\ 0 & 0 & 0 & \mathcal{A}_{44} & \mathcal{A}_{45}\\ 0 & 0 & 0 & 0 & \mathcal{A}_{55}\end{bmatrix}\mathcal{Y}\\
\tilde{E} &= \mathcal{X}\begin{bmatrix} I & 0 & 0 & 0 & \mathcal{E}_{15}\\ 0 & N_{22} & 0 & 0 & \mathcal{E}_{25}\\ 0 & \mathcal{E}_{32} & I & 0 & 0\\ 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & \mathcal{E}_{55}\end{bmatrix}\mathcal{Y}\\
\tilde{B} &= \mathcal{X}\begin{bmatrix} B_1 & 0\\ B_2 & 0\\ \mathcal{B}_{31} & \mathcal{B}_{32}\\ 0 & \mathcal{B}_{42}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{L}\\ P_L^\top\end{bmatrix}\\
\tilde{K} &= \begin{bmatrix}\hat{L}^{(+)} & P_L\end{bmatrix}\begin{bmatrix} K_1 & K_2 & 0 & 0 & \mathcal{K}_{15}\\ \mathcal{K}_{21} & \mathcal{K}_{22} & \mathcal{K}_{23} & \mathcal{K}_{24} & \mathcal{K}_{25}\end{bmatrix}\mathcal{Y}
\end{aligned} \tag{58}$$

such that matrices $\mathcal{A}_{15}, \mathcal{E}_{15} \in \mathbf{R}^{\tau_1\times\tilde{\tau}_5}$, $\mathcal{A}_{25}, \mathcal{E}_{25} \in \mathbf{R}^{\tau_2\times\tilde{\tau}_5}$, $\mathcal{A}_{31} \in \mathbf{R}^{\tilde{\tau}_3\times\tau_1}$, $\mathcal{A}_{32}, \mathcal{E}_{32} \in \mathbf{R}^{\tilde{\tau}_3\times\tau_2}$, $\mathcal{A}_{33} \in \mathbf{R}^{\tilde{\tau}_3\times\tilde{\tau}_3}$, $\mathcal{A}_{34} \in \mathbf{R}^{\tilde{\tau}_3\times\tilde{\tau}_4}$, $[\mathcal{A}_{35}^\top\ \mathcal{A}_{45}^\top]^\top \in \mathbf{R}^{(\tilde{\tau}_3+\tilde{\tau}_4)\times\tilde{\tau}_5}$, $\mathcal{A}_{55}, \mathcal{E}_{55} \in \mathbf{R}^{\tilde{\tau}_5\times\tilde{\tau}_5}$, $[\mathcal{K}_{21}\ \mathcal{K}_{22}\ \mathcal{K}_{23}\ \mathcal{K}_{24}\ \mathcal{K}_{25}] \in \mathbf{R}^{(\tilde{m}-m)\times\tilde{m}}$ and $\mathcal{K}_{15} \in \mathbf{R}^{m\times\tilde{\tau}_5}$ are arbitrary, $\tilde{\tau}_3$, $\tilde{\tau}_4$ and $\tilde{\tau}_5$ are any arbitrary non-negative integers satisfying $\tau_1+\tau_2+\tilde{\tau}_3+\tilde{\tau}_4+\tilde{\tau}_5 = \tilde{n}$, $\tau_3 \le \tilde{\tau}_5$, the pencil $s\mathcal{E}_{55}-\mathcal{A}_{55}$ is regular, $\mathcal{X}$ is any arbitrary nonsingular matrix, $\mathcal{Y}$ is any nonsingular matrix of the form

$$\mathcal{Y} = \begin{bmatrix} I & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0\\ 0 & 0 & \Upsilon_{33} & \Upsilon_{34} & \Upsilon_{35}\\ 0 & 0 & \Upsilon_{43} & \Upsilon_{44} & \Upsilon_{45}\\ 0 & 0 & \Upsilon_{53} & \Upsilon_{54} & \Upsilon_{55}\end{bmatrix}\begin{bmatrix} Y^{-1}\hat{V}\\ P_V^\top\end{bmatrix} \tag{59}$$

with any arbitrary $[\Upsilon_{33}^\top\ \Upsilon_{43}^\top\ \Upsilon_{53}^\top]^\top \in \mathbf{R}^{\sum_{i=3}^5\tilde{\tau}_i\times\tau_3}$, $\begin{bmatrix}\Upsilon_{34} & \Upsilon_{35}\\ \Upsilon_{44} & \Upsilon_{45}\\ \Upsilon_{54} & \Upsilon_{55}\end{bmatrix} \in \mathbf{R}^{\sum_{i=3}^5\tilde{\tau}_i\times(\sum_{i=3}^5\tilde{\tau}_i-\tau_3)}$, $\mathrm{r}(\Upsilon_{53}) = \tau_3$ and $\mathrm{r}[\Upsilon_{54}\ \Upsilon_{55}] = \tilde{\tau}_5-\tau_3$, and matrices $[\hat{V}^\top\ P_V]$ and $[\hat{L}^\top\ P_L]$ are any nonsingular matrices with $\hat{V}P_V = 0$, $\hat{L}P_L = 0$, $P_V^\top P_V = I$ and $P_L^\top P_L = I$.

Proof: The proof is similar to that of Theorem 1 and is omitted.

D. An Important Case: System $\mathbf{S}$ *is Controllable at Infinity*

Theorems 1–3 indicate that, in general, the matrices $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{C}, \tilde{K})$ of the system $\tilde{\mathbf{S}}$ cannot be expressed in terms of the matrices (A, E, B, C, K) of the system $\mathbf{S}$, because the existence of the block N_{33} in (9) implies that the system $\mathbf{S}$ is not controllable at infinity. To proceed, we assume controllability of the system $\mathbf{S}$ at infinity, but note that the property may be removed at the price of a more elaborate analysis. In this case, $\tau_3 = 0$, $X(sE-A)Y = \begin{bmatrix} sI-J & 0\\ 0 & sN_{22}-I\end{bmatrix}$, and Theorems 1–3 can be simplified as follows:

Theorem 4: Assume that the system $\mathbf{S}$ is controllable at infinity. Then, the system $\tilde{\mathbf{S}}$ includes the system $\mathbf{S}$ if and only if in a suitable coordinate frame we have $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{C})$ expressed by

$$\tilde{A} = \mathcal{X}\begin{bmatrix} A & 0 & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ 0 & 0 & \mathcal{A}_{33} \end{bmatrix}\begin{bmatrix} \hat{V} \\ W^\top P_V^\top \end{bmatrix}$$
$$\tilde{E} = \mathcal{X}\begin{bmatrix} E & 0 & \mathcal{E}_{13} \\ \mathcal{E}_{21} & I & \mathcal{E}_{23} \\ 0 & 0 & \mathcal{E}_{33} \end{bmatrix}\begin{bmatrix} \hat{V} \\ W^\top P_V^\top \end{bmatrix}$$
$$\tilde{B} = \mathcal{X}\begin{bmatrix} B & \mathcal{B}_{12} \\ \mathcal{B}_{21} & \mathcal{B}_{22} \\ 0 & \mathcal{B}_{32} \end{bmatrix}\begin{bmatrix} \hat{L} \\ P_L^\top \end{bmatrix}$$
$$\tilde{C} = [T \quad P_T]\begin{bmatrix} C & 0 & \mathcal{C}_{13} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} \end{bmatrix}\begin{bmatrix} \hat{V} \\ W^\top P_V^\top \end{bmatrix} \tag{60}$$

where $\mathcal{A}_{21}, \mathcal{E}_{21} \in \mathbf{R}^{\tilde{\tau}_2 \times n}$, $\mathcal{A}_{22} \in \mathbf{R}^{\tilde{\tau}_2 \times \tilde{\tau}_2}$, $\mathcal{A}_{13}, \mathcal{E}_{13} \in \mathbf{R}^{\tau_1 \times \tilde{\tau}_3}$, $\mathcal{A}_{23}, \mathcal{E}_{23} \in \mathbf{R}^{\tilde{\tau}_2 \times \tilde{\tau}_3}$, $\mathcal{A}_{33}, \mathcal{E}_{33} \in \mathbf{R}^{\tilde{\tau}_3 \times \tilde{\tau}_3}$, $[\mathcal{B}_{12}^\top \, \mathcal{B}_{22}^\top \, \mathcal{B}_{32}^\top]^\top \in \mathbf{R}^{\tilde{m} \times (\tilde{m}-m)}$, $\mathcal{B}_{21} \in \mathbf{R}^{\tilde{\tau}_2 \times m}$, $\mathcal{C}_{13} \in \mathbf{R}^{l \times \tilde{\tau}_3}$, $[\mathcal{C}_{21} \, \mathcal{C}_{22} \, \mathcal{C}_{23}] \in \mathbf{R}^{(\tilde{l}-l) \times \tilde{l}}$ are arbitrary, $\tilde{\tau}_2$ and $\tilde{\tau}_3$ are arbitrary non-negative integers satisfying $n + \tilde{\tau}_2 + \tilde{\tau}_3 = \tilde{n}$, the pencil $s\mathcal{E}_{33} - \mathcal{A}_{33}$ is regular, $\mathcal{X}$ is any nonsingular matrix and W is any arbitrary orthogonal matrix, matrices $[\hat{V}^\top \, P_V]$, $[\hat{L}^\top \, P_L]$ and $[T \, P_T]$ are any nonsingular matrices with $\hat{V}P_V = 0$, $\hat{L}P_L = 0$, $T^\top P_T = 0$, $P_V^\top P_V = I$, $P_L^\top P_L = I$ and $P_T^\top P_T = I$.

Theorem 5: Assume that the system $\mathbf{S}$ is controllable at infinity.

a) The system $\mathbf{S}$ is an aggregation of the system $\tilde{\mathbf{S}}$ if and only if in a suitable coordinate frame we have $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{C})$ expressed by

$$\tilde{A} = \mathcal{X}\begin{bmatrix} A & 0 & 0 & \mathcal{A}_{14} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} & \mathcal{A}_{24} \\ 0 & 0 & \mathcal{A}_{33} & \mathcal{A}_{34} \\ 0 & 0 & 0 & I \end{bmatrix}\begin{bmatrix} \hat{V} \\ W^\top P_V^\top \end{bmatrix}$$
$$\tilde{E} = \mathcal{X}\begin{bmatrix} E & 0 & 0 & \mathcal{E}_{14} \\ \mathcal{E}_{21} & I & \mathcal{E}_{23} & \mathcal{E}_{24} \\ 0 & 0 & \mathcal{E}_{33} & \mathcal{E}_{34} \\ 0 & 0 & 0 & \mathcal{E}_{44} \end{bmatrix}\begin{bmatrix} \hat{V} \\ W^\top P_V^\top \end{bmatrix}$$
$$\tilde{B} = \mathcal{X}\begin{bmatrix} B & 0 \\ \mathcal{B}_{21} & \mathcal{B}_{22} \\ 0 & \mathcal{B}_{32} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} L^{(+)} \\ P_L^\top \end{bmatrix}$$
$$\tilde{C} = [T \quad P_T]\begin{bmatrix} C & 0 & 0 & \mathcal{C}_{14} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} \end{bmatrix}\begin{bmatrix} \hat{V} \\ W^\top P_V^\top \end{bmatrix}$$

where $\mathcal{A}_{21}, \mathcal{E}_{21} \in \mathbf{R}^{\tilde{\tau}_2 \times n}$, $\mathcal{A}_{22} \in \mathbf{R}^{\tilde{\tau}_2 \times \tilde{\tau}_2}$, $\mathcal{A}_{23}, \mathcal{E}_{23} \in \mathbf{R}^{\tilde{\tau}_2 \times \tilde{\tau}_3}$, $\mathcal{A}_{14}, \mathcal{E}_{14} \in \mathbf{R}^{n \times \tilde{\tau}_4}$, $\mathcal{A}_{33}, \mathcal{E}_{33} \in \mathbf{R}^{\tilde{\tau}_3 \times \tilde{\tau}_3}$, $\mathcal{A}_{34}, \mathcal{E}_{34} \in \mathbf{R}^{\tilde{\tau}_3 \times \tilde{\tau}_4}$, $\mathcal{B}_{21} \in \mathbf{R}^{\tilde{\tau}_2 \times m}$, $[\mathcal{B}_{22}^\top \, \mathcal{B}_{32}^\top]^\top \in \mathbf{R}^{(\tilde{\tau}_2+\tilde{\tau}_3) \times (\tilde{m}-m)}$, $\mathcal{C}_{14} \in \mathbf{R}^{l \times \tilde{\tau}_4}$ and $[\mathcal{C}_{21} \, \mathcal{C}_{22} \, \mathcal{C}_{23} \, \mathcal{C}_{24}] \in \mathbf{R}^{(\tilde{l}-l) \times \tilde{n}}$ are arbitrary, $\tilde{\tau}_2$, $\tilde{\tau}_3$ and $\tilde{\tau}_4$ are any arbitrary non-negative integers satisfying $n + \tilde{\tau}_3 + \tilde{\tau}_4 = \tilde{n}$, the pencil $s\mathcal{E}_{33} - \mathcal{A}_{33}$ is regular, $\mathcal{E}_{44}$ is nilpotent, $\mathcal{X}$ is any nonsingular matrix, and W is any arbitrary orthogonal matrix, matrices $[\hat{V}^\top \, P_V]$, $[\hat{L}^\top \, P_L]$ and $[T \, P_T]$ are any nonsingular matrices with $\hat{V}P_V = 0$, $\hat{L}P_L = 0$, $T^\top P_T = 0$, $P_V^\top P_V = I$, $P_L^\top P_L = I$ and $P_T^\top P_T = I$.

b) The system $\mathbf{S}$ is a restriction of the system $\tilde{\mathbf{S}}$ if and only if in a suitable coordinate frame we have $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{C})$ expressed by

$$\tilde{A} = \mathcal{X}\begin{bmatrix} A & \mathcal{A}_{12} \\ 0 & \mathcal{A}_{22} \end{bmatrix}\begin{bmatrix} \hat{V} \\ P_V^\top \end{bmatrix}$$
$$\tilde{E}[B = \mathcal{X}\begin{bmatrix} E & \mathcal{E}_{12} \\ 0 & \mathcal{E}_{22} \end{bmatrix}\begin{bmatrix} \hat{V} \\ P_V^\top \end{bmatrix}$$
$$\tilde{B} = \mathcal{X}\begin{bmatrix} B & \mathcal{B}_{12} \\ 0 & \mathcal{B}_{22} \end{bmatrix}\begin{bmatrix} \hat{L} \\ P_L^\top \end{bmatrix}$$
$$\tilde{C} = [T \quad P_T]\begin{bmatrix} C & \mathcal{C}_{12} \\ 0 & \mathcal{C}_{22} \end{bmatrix}\begin{bmatrix} \hat{V} \\ P_V^\top \end{bmatrix}$$

where matrices $\mathcal{A}_{12}, \mathcal{E}_{12} \in \mathbf{R}^{n \times \tilde{\tau}_2}$, $\mathcal{A}_{22}, \mathcal{E}_{22} \in \mathbf{R}^{\tilde{\tau}_2 \times \tilde{\tau}_2}$, $[\mathcal{B}_{12}^\top \, \mathcal{B}_{22}^\top]^\top \in \mathbf{R}^{\tilde{n} \times (\tilde{m}-m)}$, $[\mathcal{C}_{12}^\top \, \mathcal{C}_{22}^\top]^\top \in \mathbf{R}^{l \times \tilde{\tau}_2}$ are arbitrary, $\tilde{\tau}_2 = \tilde{n} - n$, the pencil $s\mathcal{E}_{22} - \mathcal{A}_{22}$ is regular, and $\mathcal{X}$ is any nonsingular matrix, matrices $[\hat{V}^\top \, P_V]$, $[\hat{L}^\top \, P_L]$ and $[T \, P_T]$ are any nonsingular matrices with $\hat{V}P_V = 0$, $\hat{L}P_L = 0$, $T^\top P_T = 0$, $P_V^\top P_V = I$, $P_L^\top P_L = I$ and $P_T^\top P_T = I$.

Theorem 6: Assume that the system $\mathbf{S}$ is controllable at infinity. Then, the control law $\tilde{u} = -\tilde{K}\tilde{x} + \tilde{v}$ given for system $\tilde{\mathbf{S}}$ is contractible to the control law $u = -Kx + v$ for implementation in system $\mathbf{S}$ if and only if either $(\tilde{E}, \tilde{A}, \tilde{B})$ is given by (60), and

$$\tilde{K} = \begin{bmatrix} \hat{L}^{(+)} & P_L \end{bmatrix}\begin{bmatrix} K & 0 & \mathcal{K}_{13} \\ 0 & 0 & \mathcal{K}_{23} \end{bmatrix}\begin{bmatrix} \hat{V} \\ W^\top P_V^\top \end{bmatrix}$$

with any arbitrary $\left[\begin{smallmatrix} \mathcal{K}_{13} \\ \mathcal{K}_{23} \end{smallmatrix}\right] \in \mathbf{R}^{\tilde{m} \times \tilde{\tau}_3}$, or in a suitable coordinate frame we have $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{K})$ expressed by

$$\tilde{A} = \mathcal{X}\begin{bmatrix} A & 0 & 0 & \mathcal{A}_{14} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} & \mathcal{A}_{24} \\ 0 & 0 & \mathcal{A}_{33} & \mathcal{A}_{34} \\ 0 & 0 & 0 & \mathcal{A}_{44} \end{bmatrix}\begin{bmatrix} \hat{V} \\ W^\top P_V^\top \end{bmatrix}$$
$$\tilde{E} = \mathcal{X}\begin{bmatrix} E & 0 & 0 & \mathcal{E}_{14} \\ \mathcal{E}_{21} & I & \mathcal{E}_{23} & \mathcal{E}_{24} \\ 0 & 0 & I & \mathcal{E}_{34} \\ 0 & 0 & 0 & \mathcal{E}_{44} \end{bmatrix}\begin{bmatrix} \hat{V} \\ W^\top P_V^\top \end{bmatrix}$$
$$\tilde{B} = \mathcal{X}\begin{bmatrix} B & 0 \\ \mathcal{B}_{21} & \mathcal{B}_{22} \\ 0 & \mathcal{B}_{32} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{L} \\ P_L^\top \end{bmatrix}$$
$$\tilde{K} = [L \quad P_L]\begin{bmatrix} K & 0 & 0 & \mathcal{K}_{14} \\ \mathcal{K}_{21} & \mathcal{K}_{22} & \mathcal{K}_{23} & \mathcal{K}_{24} \end{bmatrix}\begin{bmatrix} \hat{V} \\ W^\top P_V^\top \end{bmatrix}$$

where matrices $\mathcal{A}_{14}, \mathcal{E}_{14} \in \mathbf{R}^{n \times \tilde{\tau}_4}$, $\mathcal{A}_{21}, \mathcal{E}_{21} \in \mathbf{R}^{\tilde{\tau}_2 \times n}$, $\mathcal{A}_{22} \in \mathbf{R}^{\tilde{\tau}_2 \times \tilde{\tau}_2}$, $\mathcal{A}_{23}, \mathcal{E}_{23} \in \mathbf{R}^{\tilde{\tau}_2 \times \tilde{\tau}_3}$, $[\mathcal{A}_{24}^\top \, \mathcal{A}_{34}^\top]^\top$, $[\mathcal{E}_{24}^\top \, \mathcal{E}_{34}^\top]^\top \in \mathbf{R}^{(\tilde{\tau}_2+\tilde{\tau}_3) \times \tilde{\tau}_4}$, and $\mathcal{A}_{44}, \mathcal{E}_{44} \in \mathbf{R}^{\tilde{\tau}_4 \times \tilde{\tau}_4}$ are arbitrary, $\tilde{\tau}_2$, $\tilde{\tau}_3$ and $\tilde{\tau}_4$ are any arbitrary non-negative integers satisfying $n + \tilde{\tau}_2 + \tilde{\tau}_3 + \tilde{\tau}_4 = \tilde{n}$, the pencil $s\mathcal{E}_{44} - \mathcal{A}_{44}$ is regular, $\mathcal{X}$ is any arbitrary nonsingular matrix, and W is any arbitrary orthogonal matrix, matrices $[\hat{V}^\top \, P_V]$ and $[\hat{L}^\top \, P_L]$ are any nonsingular matrices with $\hat{V}P_V = 0$, $\hat{L}P_L = 0$, $P_V^\top P_V = I$ and $P_L^\top P_L = I$.

It should be pointed out that Theorems 4–6 would be true even if the assumption that the system $\mathbf{S}$ is controllable at infinity is replaced by the assumption that the set of the orders of the poles of the system $\tilde{\mathbf{S}}$ at infinity contains the set of the orders of the poles of the system $\mathbf{S}$ at infinity.

IV. Conclusion

The main contribution of this paper is an explicit characterization (canonical form) of the expansion-contraction process and contractibility of control laws for descriptor systems. The characterization parameterizes all expansions $\tilde{\mathbf{S}}$ including the original system $\mathbf{S}$, as well as all control laws $\tilde{u} = -\tilde{K}\tilde{x} + \tilde{v}$ for expansion $\tilde{\mathbf{S}}$ contractible to the control law $u = -Kx + v$ for the contraction $\mathbf{S}$, and, thus, offers full freedom in selecting the corresponding matrices $\tilde{A}$, $\tilde{E}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{K}$ of the expansion $\tilde{\mathbf{S}}$. The presented results provide a suitable mathematical foundation for control design of descriptor systems under overlapping information structure constraints.

Appendix

A. Proof of Lemma 3

The form (10) is the Kronecker canonical form [40] of the pencil $s\tilde{E} - \tilde{A}$. The form (9) can be obtained by two steps:

- First, we compute the Kronecker canonical form [40] of the pencil $sE - A$ to get nonsingular matrices X_1 and Y_1 such that

$$X_1(sE-A)Y_1 = \begin{bmatrix} \overset{\tau_1}{sI-J} & \overset{n-\tau_1}{0} \\ 0 & sN_E - I \end{bmatrix} \begin{matrix} \}\tau_1 \\ \}n-\tau_1 \end{matrix},$$

  where J is a matrix in the Jordan canonical form and N_E is nilpotent. Denote

$$X_1 B = \begin{bmatrix} B_1 \\ B_E \end{bmatrix} \begin{matrix} \}\tau_1 \\ \}n-\tau_1 \end{matrix}.$$

- Next, we compute the controllable stair-case form [39] of the pair (N_E, B_E) to get orthogonal matrix X_2 such that

$$X_2^\top N_E X_2 = \begin{bmatrix} \overset{\tau_2}{N_{22}} & \overset{\tau_3}{N_{23}} \\ 0 & N_{33} \end{bmatrix} \begin{matrix} \}\tau_2 \\ \}\tau_3 \end{matrix}$$
$$X_2 B_E = \begin{bmatrix} B_2 \\ 0 \end{bmatrix} \begin{matrix} \}\tau_2 \\ \}\tau_3 \end{matrix}$$

  where the pair (N_{22}, B_2) is controllable.

Let

$$X = \begin{bmatrix} I & 0 \\ 0 & X_2^\top \end{bmatrix} X_1, \quad Y = Y_1 \begin{bmatrix} I & 0 \\ 0 & X_2 \end{bmatrix}.$$

Then, X and Y above provide the form (9).□

Proof of Theorem 2: We only show Part (a) since Part (b) can be proved in a similar manner.

We prove necessity first. Let the triplet (V, L, T) be such that the consistency conditions in (49) imply (50). From the proof of Lemma 4, we obtain that

$$\begin{aligned}
&\begin{bmatrix} Z_{21} \\ Z_{31} \end{bmatrix} = 0 \\
&J^i \left[Z_{11}\tilde{J} - JZ_{11} \quad B_1 L^{(+)} - Z_{11}\tilde{B}_1 \right] = 0, \quad i = 0, 1, \cdots \\
&Z^{(2)}\tilde{N}^i \tilde{B}_2 = E^{(2)} N_{22}^i B_2 L^{(+)}, \quad i = 0, 1, \cdots \\
&\left(T^{(+)}\tilde{C}_1 - C_1 Z_{11} \right) \tilde{J}^i = 0, \quad i = 0, 1, \cdots \\
&\left(T^{(+)}\hat{C}_2 - CYZ^{(2)} \right) \tilde{N}^i \tilde{B}_2 = 0, \quad i = 0, 1, \cdots.
\end{aligned}$$

The conditions above can be rewritten as

$$\begin{aligned}
&\begin{bmatrix} Z_{21} \\ Z_{31} \end{bmatrix} = 0, \quad Z_{11}\tilde{J} - JZ_{11} = 0 \\
&B_1 L^{(+)} - Z_{11}\tilde{B}_1 = 0, \quad T^{(+)}\tilde{C}_1 - C_1 Z_{11} = 0 \\
&Z^{(2)}\tilde{N}^i \tilde{B}_2 = E^{(2)} N_{22}^i B_2 L^{(+)}, \quad i = 0, 1, \cdots \\
&\left(T^{(+)}\hat{C}_2 - CYZ^{(2)} \right) \tilde{N}^i \tilde{B}_2 = 0, \quad i = 0, 1, \cdots.
\end{aligned} \tag{61}$$

Note that we have shown in the proof of Lemma 4 that the system $\tilde{\mathbf{S}}$ includes the system $\mathbf{S}$ under the transformations V, L and T if and only if (43) and (44) hold. Therefore, by using Lemma 1 we can reduce (61) to

$$\begin{aligned}
&\begin{bmatrix} \mathcal{X}_{24} \\ \mathcal{X}_{34} \end{bmatrix} = 0, \quad \hat{J}_{14} = J\mathcal{X}_{14} - \mathcal{X}_{14}\hat{J}_{44} \\
&\mathcal{B}_{12} = -\mathcal{X}_{14}\tilde{B}_{42}, \quad \hat{C}_{14} = C_1\mathcal{X}_{14}
\end{aligned} \tag{62}$$

and

$$\max_{s\in\mathbf{C}} \mathrm{r} \begin{bmatrix} sI - N_{22} & -\tilde{N}_{25} & \mathcal{B}_{22} \\ 0 & sI - \tilde{N}_{55} & \tilde{B}_{52} \\ 0 & \mathcal{X}_{15} & 0 \\ I & \mathcal{X}_{25} & 0 \\ 0 & \mathcal{X}_{35} & \\ C_2 & \hat{C}_{15} & 0 \end{bmatrix} = \tau_2 + \hat{\tau}_5. \tag{63}$$

Since

$$\max_{s\in\mathbf{C}} \mathrm{r} \begin{bmatrix} sI - N_{22} & -\tilde{N}_{25} & \mathcal{B}_{22} \\ 0 & sI - \tilde{N}_{55} & \tilde{B}_{52} \\ 0 & \mathcal{X}_{15} & 0 \\ I & \mathcal{X}_{25} & 0 \\ 0 & \mathcal{X}_{35} & \\ C_2 & \hat{C}_{15} & 0 \end{bmatrix}$$
$$= \max_{s\in\mathbf{C}} \mathrm{r} \begin{bmatrix} sI - \tilde{N}_{55} & \tilde{B}_{52} \\ N_{22}\mathcal{X}_{25} - \mathcal{X}_{25}\tilde{N}_{55} - \tilde{N}_{25} & \mathcal{B}_{22} + \mathcal{X}_{25}\tilde{B}_{52} \\ \mathcal{X}_{15} & 0 \\ \mathcal{X}_{35} & 0 \\ \hat{C}_{15} - C_2\mathcal{X}_{25} & 0 \end{bmatrix} + \tau_2$$

Lemma 1 implies that (63) holds if and only if

$$\mathcal{B}_{22} = -\mathcal{X}_{25}\tilde{B}_{52} \tag{64}$$

and

$$\max_{s\in\mathbf{C}} \mathrm{r} \begin{bmatrix} sI - \tilde{N}_{55} & \tilde{B}_{52} \\ N_{22}\mathcal{X}_{25} - \mathcal{X}_{25}\tilde{N}_{55} - \tilde{N}_{25} & 0 \\ \mathcal{X}_{15} & 0 \\ \mathcal{X}_{35} & 0 \\ \hat{C}_{15} - C_2\mathcal{X}_{25} & 0 \end{bmatrix} = \hat{\tau}_5. \tag{65}$$

By a similar derivation for (43), we have that (65) is true if and only if

$$\begin{aligned}
\hat{\mathcal{W}}^\top \tilde{N}_{55} \hat{\mathcal{W}} &= \begin{bmatrix} \overset{\hat{\tau}_5^{(1)}}{\hat{\mathcal{N}}_{55}} & \overset{\hat{\tau}_5^{(2)}}{\hat{\mathcal{N}}_{56}} \\ 0 & \hat{\mathcal{N}}_{66} \end{bmatrix} \begin{matrix} \}\tau_5^{(1)} \\ \}\hat{\tau}_5^{(2)} \end{matrix} \\
\hat{\mathcal{W}}^\top \tilde{B}_{52} &= \begin{bmatrix} \hat{\mathcal{B}}_{52} \\ 0 \end{bmatrix} \begin{matrix} \}\tau_5^{(1)} \\ \}\hat{\tau}_5^{(2)} \end{matrix} \\
\mathcal{X}_{15} \hat{\mathcal{W}} &= [\overset{\hat{\tau}_5^{(1)}}{0} \quad \overset{\hat{\tau}_5^{(2)}}{\mathcal{X}_{16}}], \quad \mathcal{X}_{35} \hat{\mathcal{W}} = [\overset{\hat{\tau}_5^{(1)}}{0} \quad \overset{\hat{\tau}_5^{(2)}}{\mathcal{X}_{36}}] \\
\tilde{N}_{25} \hat{\mathcal{W}} &= \left[\overset{\hat{\tau}_5^{(1)}}{N_{22}\mathcal{X}_{25}^{(1)} - \mathcal{X}_{25}^{(1)} \hat{\mathcal{N}}_{55}} \quad \overset{\hat{\tau}_5^{(2)}}{\hat{\mathcal{N}}_{26} - \mathcal{X}_{25}^{(1)} \hat{\mathcal{N}}_{56}} \right] \\
\hat{C}_{15} \hat{\mathcal{W}} &= \left[\overset{\hat{\tau}_5^{(1)}}{C_2 \mathcal{X}_{25}^{(1)}} \quad \overset{\hat{\tau}_5^{(2)}}{\hat{\mathcal{C}}_{16}} \right]
\end{aligned} \tag{66}$$

where $\hat{\mathcal{W}} \in \mathbf{R}^{\hat{\tau}_5 \times \hat{\tau}_5}$ is orthogonal, and

$$\mathcal{X}_{25} \hat{\mathcal{W}} = \left[\overset{\hat{\tau}_5^{(1)}}{\mathcal{X}_{25}^{(1)}} \quad \overset{\hat{\tau}_5^{(2)}}{\mathcal{X}_{26}} \right].$$

Hence, up to now we have shown that the system $\mathbf{S}$ is an aggregation of the system $\tilde{\mathbf{S}}$ only if (43), (44), (62), (64) and (66) hold, i.e.,

$$\begin{aligned}
\tilde{J} &= \mathcal{Z}_U \mathbb{X}_{134} \begin{bmatrix} J & 0 & 0 \\ \mathcal{A}_{31} & \mathcal{A}_{33} & \hat{J}_{34} \\ 0 & 0 & \hat{J}_{44} \end{bmatrix} \mathbb{X}_{134}^{-1} \mathcal{Z}_U^{-1} \\
\mathcal{Z}_U &= [\mathcal{Z}_{11} \quad U\mathcal{W}] \\
\mathbb{X}_{134} &= \begin{bmatrix} I & -\mathcal{X}_{13} & -\mathcal{X}_{14} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\
\tilde{B}_1 &= \mathcal{Z}_U \mathbb{X}_{134} \begin{bmatrix} B_1 & 0 \\ \mathcal{B}_{31} & \mathcal{B}_{32} \\ 0 & \tilde{B}_{42} \end{bmatrix} \begin{bmatrix} L^{(+)} \\ P_L^\top \end{bmatrix} \\
\tilde{C}_1 &= [T \quad P_T] \begin{bmatrix} C_1 & 0 & 0 \\ \mathcal{C}_{21} & \mathcal{C}_{23} & \hat{C}_{24} \end{bmatrix} \mathbb{X}_{134}^{-1} \mathcal{Z}_U^{-1} \\
\mathcal{Z}_{12} &= \mathcal{Z}_U \mathbb{X}_{134} \begin{bmatrix} 0 \\ \Upsilon_{32} \\ 0 \end{bmatrix} \\
Z^{(1)} &= \left[E^{(1)} \quad 0 \quad 0 \right] \mathbb{X}_{134}^{-1} \mathcal{Z}_U^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{N} &= \mathcal{Z}_{2QW} \mathbb{X}_{25} \begin{bmatrix} N_{22} & 0 & \hat{\mathcal{N}}_{26} \\ 0 & \hat{\mathcal{N}}_{55} & \hat{\mathcal{N}}_{56} \\ 0 & 0 & \hat{\mathcal{N}}_{66} \end{bmatrix} \mathbb{X}_{25}^{1} \mathcal{Z}_{2QW}^{-1} \\
\mathcal{Z}_{2QW} &= [\mathcal{Z}_{22} \quad Q\hat{\mathcal{W}}] \\
\mathbb{X}_{25} &= \begin{bmatrix} I & -\mathcal{X}_{25}^{(1)} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\
\tilde{B}_2 &= \mathcal{Z}_{2QW} \mathbb{X}_{25} \begin{bmatrix} B_2 & 0 & \\ 0 & \hat{\mathcal{B}}_{52} 0 & 0 \end{bmatrix} \begin{bmatrix} L^{(+)} \\ P_L^\top \end{bmatrix} \\
\hat{C}_2 &= [T \quad P_T] \begin{bmatrix} C_2 & 0 & \hat{\mathcal{C}}_{16} \\ \tilde{\mathcal{C}}_{22} & \hat{\mathcal{C}}_{25} & \hat{\mathcal{C}}_{26} \end{bmatrix} \mathbb{X}_{25}^{1} \mathcal{Z}_{2QW}^{-1} \\
\mathcal{Z}_{21} &= 0.
\end{aligned}$$

Obviously, since $\tilde{N}$ is nilpotent, $\hat{\mathcal{N}}_{66}$ is nilpotent. As a result, we conclude that the system $\mathbf{S}$ is an aggregation of the system $\tilde{\mathbf{S}}$ only if

$$\begin{aligned}
\tilde{A} &= \mathcal{X} \begin{bmatrix} J & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ \mathcal{A}_{31} & 0 & \mathcal{A}_{33} & \hat{J}_{34} & 0 & 0 \\ 0 & 0 & 0 & \hat{J}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \mathcal{Y} \\
&= \mathcal{X} \begin{bmatrix} J & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ \mathcal{A}_{31} & 0 & \mathcal{A}_{33} & \mathcal{A}_{34} & 0 \\ 0 & 0 & 0 & \mathcal{A}_{44} & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \mathcal{Y}
\end{aligned}$$

$$\mathcal{A}_{34} := [\hat{J}_{34} \quad 0], \quad \mathcal{A}_{44} := \begin{bmatrix} \hat{J}_{44} & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{aligned}
\tilde{E} &= \mathcal{X} \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & N_{22} & 0 & 0 & 0 & \hat{\mathcal{N}}_{26} \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{\mathcal{N}}_{55} & \hat{\mathcal{N}}_{56} \\ 0 & 0 & 0 & 0 & 0 & \hat{\mathcal{N}}_{66} \end{bmatrix} \mathcal{Y} \\
&= \mathcal{X} \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & N_{22} & 0 & 0 & \tilde{\mathcal{E}}_{25} \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & \mathcal{E}_{44} & \mathcal{E}_{45} \\ 0 & 0 & 0 & 0 & \mathcal{E}_{55} \end{bmatrix} \mathcal{Y}
\end{aligned}$$

$$\tilde{\mathcal{E}}_{25} := \hat{\mathcal{N}}_{26}, \quad \mathcal{E}_{44} := \begin{bmatrix} I & 0 \\ 0 & \hat{\mathcal{N}}_{55} \end{bmatrix}, \quad \mathcal{E}_{45} := \begin{bmatrix} 0 \\ \hat{\mathcal{N}}_{56} \end{bmatrix}$$

$$\mathcal{E}_{55} := \hat{\mathcal{N}}_{66}$$

$$\begin{aligned}
\tilde{B} &= \mathcal{X} \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \\ \mathcal{B}_{31} & \mathcal{B}_{32} \\ 0 & \tilde{B}_{42} \\ 0 & \hat{\mathcal{B}}_{52} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} L^{(+)} \\ P_L^\top \end{bmatrix} \\
&= \mathcal{X} \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \\ \mathcal{B}_{31} & \mathcal{B}_{32} \\ 0 & \mathcal{B}_{42} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} L^{(+)} \\ P_L^\top \end{bmatrix}, \quad \mathcal{B}_{42} := \begin{bmatrix} \tilde{B}_{42} \\ \hat{\mathcal{B}}_{52} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\tilde{C} &= [T \quad P_T] \begin{bmatrix} C_1 & C_2 & 0 & 0 & 0 & \hat{\mathcal{C}}_{16} \\ \mathcal{C}_{21} & \tilde{\mathcal{C}}_{22} & \mathcal{C}_{23} & \hat{C}_{24} & \hat{\mathcal{C}}_{25} & \hat{\mathcal{C}}_{26} \end{bmatrix} \mathcal{Y} \\
&= [T \quad P_T] \begin{bmatrix} C_1 & C_2 & 0 & 0 & \tilde{\mathcal{C}}_{15} \\ \mathcal{C}_{21} & \tilde{\mathcal{C}}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} & \tilde{\mathcal{C}}_{25} \end{bmatrix} \mathcal{Y}
\end{aligned}$$

$$\tilde{\mathcal{C}}_{15} := \hat{\mathcal{C}}_{16}, \quad \mathcal{C}_{24} := [\hat{C}_{24} \quad \hat{\mathcal{C}}_{25}], \quad \tilde{\mathcal{C}}_{25} := \hat{\mathcal{C}}_{26}$$

with $\tilde{\tau}_5 = \hat{\tau}_5^{(2)}$.

$$\mathcal{X} = \tilde{X}^{-1} \mathcal{Z}_{12U} \hat{\mathbb{X}}_{345}^{-1}$$

and

$$\mathcal{Y} = \hat{\mathbb{X}}_{345} \mathcal{Z}_{12U}^{-1} \tilde{Y}^{-1}.$$

where

$$\mathcal{Z}_{12U} = \begin{bmatrix} \mathcal{Z}_{11} & U\mathcal{W} & 0 & 0 \\ 0 & 0 & \mathcal{Z}_{22} & Q\hat{\mathcal{W}} \end{bmatrix}$$

$$\hat{\mathbb{X}}_{345} = \begin{bmatrix} I & \mathcal{X}_{13} & \mathcal{X}_{14} & 0 & 0 & 0 \\ 0 & 0 & 0 & I & \mathcal{X}_{25}^{(1)} & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

Since

$$Y^{-1}V^{(+)}\tilde{Y} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & \mathcal{X}_{16} \\ 0 & I & 0 & 0 & 0 & \mathcal{X}_{26} \\ 0 & 0 & 0 & 0 & 0 & \mathcal{X}_{36} \end{bmatrix} \hat{\mathbb{X}}_{345}\mathcal{Z}_{12U}^{-1}$$

and

$$\tilde{Y}^{-1}VY = \mathcal{Z}_{12U}\hat{\mathbb{X}}_{345}^{-1} \begin{bmatrix} I & 0 & \Upsilon_{13} \\ 0 & I & \Upsilon_{23} \\ 0 & \Upsilon_{32} & \Upsilon_{33} \\ 0 & 0 & \hat{\Upsilon}_{43} \\ 0 & 0 & \hat{\Upsilon}_{53} \\ 0 & 0 & \hat{\Upsilon}_{63} \end{bmatrix}, \quad \begin{bmatrix} \Upsilon_{13} \\ \Upsilon_{23} \\ \Upsilon_{33} \\ \hat{\Upsilon}_{43} \\ \hat{\Upsilon}_{53} \\ \hat{\Upsilon}_{63} \end{bmatrix} = \begin{bmatrix} \mathcal{Z}_{13} \\ \mathcal{Z}_{23} \end{bmatrix},$$

we get

$$\begin{aligned} V^{(+)} &= Y \begin{bmatrix} I & 0 & 0 & 0 & 0 & \mathcal{X}_{16} \\ 0 & I & 0 & 0 & 0 & \mathcal{X}_{26} \\ 0 & 0 & 0 & 0 & 0 & \mathcal{X}_{36} \end{bmatrix} \tilde{\mathcal{Y}} \\ &= Y \begin{bmatrix} I & 0 & 0 & 0 & \Theta_{15} \\ 0 & I & 0 & 0 & \Theta_{25} \\ 0 & 0 & 0 & 0 & \Theta_{35} \end{bmatrix} \tilde{\mathcal{Y}}, \quad \begin{bmatrix} \Theta_{15} \\ \Theta_{25} \\ \Theta_{35} \end{bmatrix} = \begin{bmatrix} \mathcal{X}_{16} \\ \mathcal{X}_{26} \\ \mathcal{X}_{36} \end{bmatrix} \end{aligned}$$

$$\tilde{\mathcal{Y}}VY = \begin{bmatrix} I & 0 & \Upsilon_{13} \\ 0 & I & \Upsilon_{23} \\ 0 & \Upsilon_{32} & \Upsilon_{33} \\ 0 & 0 & \hat{\Upsilon}_{43} \\ 0 & 0 & \hat{\Upsilon}_{53} \\ 0 & 0 & \hat{\Upsilon}_{63} \end{bmatrix} = \begin{bmatrix} I & 0 & \Upsilon_{13} \\ 0 & I & \Upsilon_{23} \\ 0 & \Upsilon_{32} & \Upsilon_{33} \\ 0 & 0 & \Upsilon_{43} \\ 0 & 0 & \Upsilon_{53} \end{bmatrix}$$

where

$$\Upsilon_{43} = \begin{bmatrix} \hat{\Upsilon}_{43} \\ \hat{\Upsilon}_{53} \end{bmatrix}, \quad \Upsilon_{53} = \hat{\Upsilon}_{63}.$$

Using the property $V^{+}V = I$ we obtain

$$\tilde{\mathcal{Y}} = \begin{bmatrix} I & 0 & 0 & 0 & -\Theta_{15} \\ 0 & I & 0 & 0 & -\Theta_{25} \\ 0 & \Upsilon_{32} & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & \Upsilon_{33} & \Upsilon_{34} & \Upsilon_{35} \\ 0 & 0 & \Upsilon_{43} & \Upsilon_{44} & \Upsilon_{45} \\ 0 & 0 & \Upsilon_{53} & \Upsilon_{54} & \Upsilon_{55} \end{bmatrix} \times \begin{bmatrix} Y^{-1}V^{(+)} \\ P_V^{\top} \end{bmatrix}$$

and

$$\mathrm{r}(\Upsilon_{53}) = \tau_3, \quad \mathrm{r}[\Upsilon_{54} \quad \Upsilon_{55}] = \tilde{\tau}_5 - \tau_3.$$

Therefore, (52) and (53) hold with

$$\begin{aligned} &\mathcal{A}_{15} = -J\Theta_{15}, \quad \mathcal{A}_{25} = -\Theta_{25} \\ &\mathcal{A}_{32} = \mathcal{A}_{33}\Upsilon_{32}, \quad \mathcal{A}_{35} = -\mathcal{A}_{31}\Theta_{15} \\ &\mathcal{E}_{15} = -\Theta_{15}, \quad \mathcal{E}_{25} = \tilde{\mathcal{E}}_{25} - N_{22}\Theta_{25}, \quad \mathcal{E}_{32} = \Upsilon_{32} \\ &\mathcal{C}_{15} = \tilde{\mathcal{C}}_{15} - C_1\Theta_{15} - C_2\Theta_{25} \\ &\mathcal{C}_{22} = \tilde{\mathcal{C}}_{22} + \mathcal{C}_{23}\Upsilon_{32}, \quad \mathcal{C}_{25} = \tilde{\mathcal{C}}_{25} - \mathcal{C}_{21}\Theta_{15} - \tilde{\mathcal{C}}_{22}\Theta_{25} \end{aligned}$$

and

$$\hat{V} = V^{(+)}, \quad \hat{L} = L^{(+)}.$$

Conversely, let the quadruplet $(\tilde{A}, \tilde{E}, \tilde{B}, \tilde{C})$ be given by (52) and (53), and let $V = \hat{V}^{+}$ and $L = \hat{L}^{+}$. Then, a direct verification gives that for any fixed input $\tilde{u}(t)$ and any consistent initial state $\tilde{x}^0$ of system $\tilde{\mathbf{S}}$, the consistency conditions in (49) imply (50). Hence, the system $\mathbf{S}$ is an aggregation of the system $\tilde{\mathbf{S}}$.□

B. *Proof of Lemma 6*

As in the proof of Lemma 4, we have that one of the statements (a) and (b) in Definition 2 holds true under the transformations V, L and T if and only if either

$$\begin{aligned} &Z^{(1)}\mathcal{Z}_{11} = E^{(1)}, \quad Z^{(2)}\mathcal{Z}_{22} = E^{(2)}, \quad \mathcal{Z}_{21} = 0 \\ &Z^{(1)}\tilde{J}^i[\tilde{J}\mathcal{Z}_{11} - \mathcal{Z}_{11}J \quad \tilde{B}_1L - \mathcal{Z}_{11}B_1 \quad \mathcal{Z}_{12}] = 0 \\ &\quad i = 0, 1, \cdots \\ &\hat{N}^i\tilde{B}_2L = \mathcal{Z}_{22}N_{22}^iB_2, \quad i = 0, 1, \cdots \\ &(\tilde{K}_1 - LK_1Z_{11})\tilde{J}^i[\mathcal{Z}_{11} \quad \tilde{B}_1L - \mathcal{Z}_{11}B_1 \quad \mathcal{Z}_{12}] = 0 \\ &\quad i = 0, 1, \cdots \\ &\hat{K}_2\mathcal{Z}_{22} = LK_2 \end{aligned}$$

or

$$\begin{aligned} &Z^{(1)}\mathcal{Z}_{11} = E^{(1)}, \quad Z^{(2)}\mathcal{Z}_{22} = E^{(2)}, \quad \mathcal{Z}_{21} = 0 \\ &Z^{(2)}\tilde{J}^i\left[\tilde{J}\mathcal{Z}_{11} - \mathcal{Z}_{11}J \quad \tilde{B}_1 - \mathcal{Z}_{11}B_1L^{(+)} \quad \mathcal{Z}_{12}\right] = 0 \\ &\quad i = 0, 1, \cdots \\ &\hat{N}^i\tilde{B}_2 = \mathcal{Z}_{22}N_{22}^iB_2L^{(+)}, \quad i = 0, 1, \cdots \\ &\left(L^{(+)}\tilde{K}_1 - K_1Z_{11}\right)\tilde{J}^i \\ &\quad \times \left[\mathcal{Z}_{11} \quad \tilde{B}_1 - \mathcal{Z}_{11}B_1L^{(+)} \quad \mathcal{Z}_{12}\right] = 0, \quad i = 0, 1, \cdots \\ &L^{(+)}\hat{K}_2\mathcal{Z}_{22} = K_2. \end{aligned}$$

Therefore, Lemma 6 follows directly from Lemma 1. □

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