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Stabilization of fixed modes in expansions of LTI systems

Srdjan S. Stanković[a,*], Dragoslav D. Šiljak[b]

[a] *Faculty of Electrical Engineering, University of Belgrade, Belgrade, Serbia*
[b] *School of Engineering, Santa Clara University, Santa Clara, CA, USA*

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Abstract

The objective of this note is to propose a method for stabilization of structurally fixed modes in expansions of LTI dynamic systems in the scope of overlapping decentralized control design based on the expansion/contraction framework, enabling successful design for a broader class of problems than considered so far. The method is based on a judicious choice of complementary matrices in the expanded space. Numerical examples are provided to illustrate simplicity and efficiency of the proposed approach.

Keywords: Inclusion principle; Expansions of LTI systems; Stabilization; Decentralized overlapping control; Controller contractibility

1. Introduction

It has been often found advantageous, for either conceptual or computational reasons, to decompose large complex systems into overlapping subsystems sharing common parts, and to apply decentralized control strategies which offer satisfactory performance at minimal communication cost [14,18,9,10,13,1,7,19]. To design the control law, the designer *expands* the system into a larger space where the *subsystems* appear as disjoint, designs *decentralized controllers* in the expanded space using standard methods, and then *contracts* the local controllers to the original space for implementation. The mathematical framework for expansion and contraction has become known as the *inclusion principle* [16,18], and has been applied in fields as diverse as electric power systems [20], mechanical structures [1], applied mathematics [18,2], automated highway systems [23] and formations of unmanned arial vehicles [24].

One of the obstacles in the design of decentralized overlapping control with information structure constraints is the presence of *structurally fixed modes* in the expanded models [17,14,18]. It has been found that *instability* of these modes can cause a conflict between *contractibility* and *stability* requirements. Either contractibility of the expanded control law is guaranteed but stabilizability is not due to unstable fixed modes in the expanded space or stabilizability is achieved but the control law is not contractible to the original space [1–4,9,10,12–16,20–24]. One of the references that discusses the expansion/contraction paradigm in this context is the book [18], which contains a tutorial-like presentation of the structurally fixed mode problem in a two-area power system interconnected by a tie-line (Section 8.3; in particular, Examples 8.17 and 8.22). Recently, an LMI approach has been proposed [25], which addresses the joint contractibility–stability problem at the price of numerical difficulties in computing design parameters involving stabilizing gain matrices.

The objective of this note is to demonstrate that the contractibility–stability dilemma can be resolved by choosing appropriately the complementary matrices at the outset of the expansion. It will be shown that the proposed method represents a simple and efficient tool for overlapping decentralized control design using *restriction* (*extension*) [9,10,12,13,8] within the LMI framework. The proposed approach is also in the spirit of the work in [2–4], exploiting flexibility of complementary matrices to achieve a desired overlapping structure for control system design.

2. Overlapping decompositions: a generalization

Consider a linear time-invariant (LTI) dynamic system with the state model

$$\mathbf{S}: \ \dot{x} = Ax + Bu, \quad x(t_0) = x_0, \tag{1}$$

* Corresponding author. Tel.: +381 11 3370150; fax: +381 11 3248681.
E-mail address: stankovic@etf.bg.ac.yu (S.S. Stanković).

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in which matrices $A = (A_{ij})$ and $B = (B_{ij})$ are decomposed into compatible block-matrices A_{ij} and B_{ij} with dimensions $(n_i \times n_j)$ and $(n_i \times m_j)$, respectively $(i, j = 1, 2, \ldots, N)$; accordingly, the state and input vectors x and u can be represented as $x = [x_1^T x_2^T \ldots x_N^T]^T$ and $u = [u_1^T u_2^T \ldots u_N^T]^T$, respectively, with dim $x_i = n_i$, dim $u_i = m_i$, $\sum_{i=1}^N n_i = n$, $\sum_{i=1}^N m_i = m$. The assumed structure of $\mathbf{S}$ can be induced either by specific structural properties of a real system or as an artifice designed to improve the efficiency of analysis/synthesis methods applied to $\mathbf{S}$.

The *expansion* $\tilde{\mathbf{S}}$ of $\mathbf{S}$ is defined by

$$\tilde{\mathbf{S}}: \; \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}, \quad \tilde{x}(t_0) = \tilde{x}_0, \tag{2}$$

where $\tilde{x}$ (dim $\tilde{x} = \tilde{n} \geqslant n$) and $\tilde{u}$ (dim $\tilde{u} = \tilde{m} \geqslant m$) are the expanded state and input vectors, respectively. It is assumed that the expansion $\tilde{\mathbf{S}}$ *includes* the original system $\mathbf{S}$ according to the inclusion principle ([16], for a comprehensive introductory treatment see [18]). This property implies that the states and inputs of $\mathbf{S}$ and $\tilde{\mathbf{S}}$ are related by full rank linear transformations $V_{\tilde{n}\times n}$, $U_{n\times\tilde{n}}$, $R_{\tilde{m}\times m}$ and $Q_{m\times\tilde{m}}$ satisfying $UV = I$ and $QR = I$, in such a way that $\tilde{x} = Vx$ (or $x = U\tilde{x}$) and $\tilde{u} = Ru$ (or $u = Q\tilde{u}$). As in the case of inclusion all the motions of $\mathbf{S}$ are included in $\tilde{\mathbf{S}}$, stability of $\tilde{\mathbf{S}}$ implies stability of $\mathbf{S}$.

We shall consider in the sequel expansions satisfying *restriction* conditions, having in mind that restriction, as a special type of inclusion, has been found to be the most suitable for control design [14,18,20,23,24]. The basic property of restriction is that $\tilde{x}(t) = Vx(t)$ for all $t \geqslant t_0$; the corresponding *inclusion condition* relating parameters of $\mathbf{S}$ and $\tilde{\mathbf{S}}$ is $\tilde{A}V = VA$ (see e.g. [18,20,22]). Solving the last equation for $\tilde{A}$, we obtain $\tilde{A} = VAU + M$, where M is a *complementary matrix* given by

$$M = Y(I - VU) = Y\hat{V}\hat{U}, \tag{3}$$

where Y is any $\tilde{n} \times \tilde{n}$ matrix, $\hat{V}$ is any basis matrix for $\mathcal{N}(U)$ and $\hat{U}$ is the unique left inverse of $\hat{V}$ such that $\mathcal{N}(\hat{U}) = \mathcal{R}(V)$ ($\mathcal{N}(.)$ indicates the null space and $\mathcal{R}(.)$ the range space of an indicated matrix) [5,18,16]. Consequently, the condition $\tilde{A}V = VA$ is equivalent to $MV = 0$ [18,16]. Inputs of $\tilde{\mathbf{S}}$ and $\mathbf{S}$ can be related either by $\tilde{u} = Ru$ or by $u = Q\tilde{u}$ [18,16]. In the first case, we have restriction, type (a), with the additional inclusion condition $\tilde{B}R = VB$ related to the inputs, and in the second case restriction, type (b), (or *extension* [9,10,12,13]) with the additional condition $\tilde{B} = VBQ$ [15,18,22].

It is very important for our further discussion to remark that the inclusion of $\mathbf{S}$ by $\tilde{\mathbf{S}}$ implies

$$\pi(\tilde{A}) = \pi(A)\pi(\hat{U}M\hat{V}) = \pi(A)\pi(\hat{U}Y\hat{V}), \tag{4}$$

where $\pi(.)$ denotes the characteristic polynomial of an indicated matrix [16,18], i.e. the modes introduced in $\tilde{\mathbf{S}}$ by expansion are the eigenvalues of $\hat{U}M\hat{V}$ (or $\hat{U}Y\hat{V}$).

To broaden the scope of inclusion principle in *overlapping decentralized control* design, we shall consider the case of restriction using a general form of state transformation matrix $V = \left[V_1^T \vdots V_2^T \cdots \vdots V_\mu^T \right]^T$, compatible with the assumed decomposition of A and B in (1). Each block V_i $(i = 1, \ldots, \mu)$ is characterized by a positive integer N_i $(N_i \in \{1, \ldots, N\})$ and positive integers $l_1^i, l_2^i, \ldots, l_{N_i}^i$ $(l_k^i \in \{1, \ldots, N\})$ in such a way that $V_i = \left[v_1^{iT} \vdots \cdots \vdots v_{N_i}^{iT} \right]^T$, where v_k^{iT} $(k = 1, \ldots, N_i)$ is a block-row matrix of the form $v_k^i = \left[0 \vdots \cdots \vdots I_{n_{l_k^i} \times n_{l_k^i}} \vdots \cdots \vdots 0 \right]$, in which $I_{n_{l_k^i} \times n_{l_k^i}}$ is located at the l_k^ith block column index (obviously, v_i^k is an $n_{l_i^k} \times n$ matrix). This fact implies that the state vector of $\tilde{\mathbf{S}}$ is $\left[x_{l_1^1}^T \cdots x_{l_{N_1}^1}^T \vdots \cdots \vdots x_{l_1^\mu}^T \cdots x_{l_{N_\mu}^\mu}^T \right]^T$. Input expansion can be done in different ways, depending on the structure of B and the aims of the design; if it is assumed (logically) to have a compatible structure with V, we must have $\tilde{u} = \left[u_{l_1^1}^T \cdots u_{l_{N_1}^1}^T \vdots \cdots \vdots u_{l_1^\mu}^T \cdots u_{l_{N_\mu}^\mu}^T \right]^T$. Consequently, the obtained expansion $\tilde{\mathbf{S}}$ can be viewed as an interconnection of μ *overlapping subsystems* of $\mathbf{S}$, represented for $i = 1, \ldots, \mu$ by

$$\tilde{\mathbf{S}}^i: \; \dot{\tilde{x}}^i = \tilde{A}^i\tilde{x}^i + \tilde{B}^i\tilde{u}^i, \quad \tilde{x}^i(t_0) = \tilde{x}_0^i, \tag{5}$$

where $\tilde{x}^i = [x_{l_1^i}^T x_{l_2^i}^T \ldots x_{l_{N_i}^i}^T]^T$, $\tilde{u}^i = [u_{l_1^i}^T u_{l_2^i}^T \ldots u_{l_{N_i}^i}^T]^T$,

$$\tilde{A}^i = \begin{bmatrix} A_{l_1^i l_1^i} & A_{l_1^i l_2^i} & \cdots & A_{l_1^i l_{N_i}^i} \\ A_{l_2^i l_1^i} & A_{l_2^i l_2^i} & \cdots & A_{l_2^i l_{N_i}^i} \\ & \cdots & & \\ A_{l_{N_i}^i l_1^i} & A_{l_{N_i}^i l_2^i} & \cdots & A_{l_{N_i}^i l_{N_i}^i} \end{bmatrix},$$

$$\tilde{B}^i = \begin{bmatrix} B_{l_1^i l_1^i} & B_{l_1^i l_2^i} & \cdots & B_{l_1^i l_{N_i}^i} \\ B_{l_2^i l_1^i} & B_{l_2^i l_2^i} & \cdots & B_{l_2^i l_{N_i}^i} \\ & \cdots & & \\ B_{l_{N_i}^i l_1^i} & B_{l_{N_i}^i l_2^i} & \cdots & B_{l_{N_i}^i l_{N_i}^i} \end{bmatrix} \tilde{R}^i$$

($|\tilde{R}^i| \neq 0$). Consequently, matrices $\tilde{A}$ and $\tilde{B}$ in the expansion $\tilde{\mathbf{S}}$ contain all the matrices $\tilde{A}^i$ and $\tilde{B}^i$ $(i = 1, \ldots, \mu)$ as their constituent diagonal blocks; the remaining elements of $\tilde{A}$ and $\tilde{B}$, which correspond to the interconnections between the subsystems $\tilde{\mathbf{S}}^i$, can be selected in different ways, but always in accordance with the inclusion principle. Extracting subsystems $\tilde{\mathbf{S}}^i$ from $\tilde{\mathbf{S}}$ one obtains different *overlapping decompositions* of $\mathbf{S}$. To illustrate the bookkeeping involved in the proposed general expansion scheme, we use the following example:

Example 1. Assuming $N = 3$, $\mu = 3$, $N_1 = N_2 = N_3 = 2$, with $l_1^1 = 1$, $l_2^1 = 2$, $l_1^2 = 1$, $l_2^2 = 3$, $l_1^3 = 2$ and $l_2^3 = 3$, we have $V = \begin{bmatrix} I & 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & I \end{bmatrix}^T$. For a full block-matrix $A = (A_{ij})$

we obtain the following state matrix in the expansion (2)

$$\tilde{A}=\left[\begin{array}{cc:cc:cc} A_{11} & A_{12} & 0 & A_{13} & 0 & 0\\ A_{21} & A_{22} & 0 & 0 & 0 & A_{23}\\ \hdashline 0 & A_{12} & A_{11} & A_{13} & 0 & 0\\ 0 & 0 & A_{31} & A_{33} & A_{32} & 0\\ \hdashline A_{21} & 0 & 0 & 0 & A_{22} & A_{23}\\ 0 & 0 & A_{31} & 0 & A_{32} & A_{33} \end{array}\right] \tag{6}$$

using $\tilde{A}=VAU+M$, where $U=V^{+}$ (V^{+} denotes the pseudoinverse of V), with

$$VAU=\frac{1}{2}\left[\begin{array}{cc:cc:cc} A_{11} & A_{12} & A_{11} & A_{13} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{21} & A_{23} & A_{22} & A_{23}\\ \hdashline A_{11} & A_{12} & A_{11} & A_{13} & A_{12} & A_{13}\\ A_{31} & A_{32} & A_{31} & A_{33} & A_{32} & A_{33}\\ \hdashline A_{21} & A_{22} & A_{21} & A_{23} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{31} & A_{33} & A_{32} & A_{33} \end{array}\right]$$

and

$$M=\frac{1}{2}\left[\begin{array}{cc:cc:cc} A_{11} & A_{12} & -A_{11} & A_{13} & -A_{12} & -A_{13}\\ A_{21} & A_{22} & -A_{21} & -A_{23} & -A_{22} & A_{23}\\ \hdashline -A_{11} & A_{12} & A_{11} & A_{13} & -A_{12} & -A_{13}\\ -A_{31} & -A_{32} & A_{31} & A_{33} & A_{32} & -A_{33}\\ \hdashline A_{21} & -A_{22} & -A_{21} & -A_{23} & A_{22} & A_{23}\\ -A_{31} & -A_{32} & A_{31} & -A_{33} & A_{32} & A_{33} \end{array}\right]$$

Assuming $B=\operatorname{diag}\{B_{11},B_{22},B_{33}\}$ and using R in the same form as V, we obtain, in the case of restriction, type (a), $\tilde{B}=\operatorname{diag}\{B_{11},B_{22},B_{11},B_{33},B_{22},B_{33}\}$ from $\tilde{B}R=VB$, and in the case of restriction, type (b) (extension),

$$\tilde{B}=VBQ=\frac{1}{2}\left[\begin{array}{cc:cc:cc} B_{11} & 0 & B_{11} & 0 & 0 & 0\\ 0 & B_{22} & 0 & 0 & B_{22} & 0\\ \hdashline B_{11} & 0 & B_{11} & 0 & 0 & 0\\ 0 & 0 & 0 & B_{33} & 0 & B_{33}\\ \hdashline 0 & B_{22} & 0 & 0 & B_{22} & 0\\ 0 & 0 & 0 & B_{33} & 0 & B_{33} \end{array}\right]; \tag{7}$$

in the first case, $\tilde{R}^{i}=I$, and in the second, $\tilde{R}^{i}=\frac{1}{2}I$, according to (5). Dotted lines in (6) and (7) delineate the overlapping subsystems of the form (5).

Example 2. The generic case, which has been treated in numerous papers (e.g. [1,9,14,15,18]), can be easily constructed on the basis of Example 1, taking $N=3$, $\mu=2$, $N_1=N_2=2$, $l_1^1=1$, $l_2^1=2$, $l_1^2=2$, $l_2^2=3$, $V^T=\left[\begin{array}{cc:cc} I & 0 & 0 & 0\\ 0 & I & I & 0\\ 0 & 0 & 0 & I \end{array}\right]$ and $R^T=\left[\begin{array}{cc:cc} I & 0 & 0 & 0\\ 0 & I & I & 0\\ 0 & 0 & 0 & I \end{array}\right]$.

Then, we can obtain similarly as above

$$\tilde{A}=\left[\begin{array}{cc:cc} A_{11} & A_{12} & 0 & A_{13}\\ A_{21} & A_{22} & 0 & A_{23}\\ \hdashline A_{21} & 0 & A_{22} & A_{23}\\ A_{31} & 0 & A_{32} & A_{33} \end{array}\right],\quad \tilde{B}=\frac{1}{2}\left[\begin{array}{cc:cc} B_{11} & 0 & 0 & 0\\ 0 & B_{22} & B_{22} & 0\\ \hdashline 0 & B_{22} & B_{22} & 0\\ 0 & 0 & 0 & B_{33} \end{array}\right]$$

in case of restriction, type (b) [9,18].

Decomposition of the original system (1) into overlapping subsystems (5) enables us to design a *decentralized control strategy* for the generated expansion (2). In the standard control design (e.g. [18,22]), the local state feedback control laws for the overlapping subsystems $\tilde{\mathbf{S}}^i$ are given by $\tilde{u}_i=\tilde{K}_i\tilde{x}_i$, where $\tilde{K}_i$ are $m_i\times n_i$ constant matrices. Consequently, for the whole expansion $\tilde{\mathbf{S}}$ we adopt the structure $\tilde{u}=\tilde{K}_D\tilde{x}$, where $\tilde{K}_D=\operatorname{diag}\{\tilde{K}_1,\dots,\tilde{K}_N\}$. Design of stabilizing $\tilde{K}_D$ for $\tilde{\mathbf{S}}$ can be done in numerous ways, including strategies based on the application of LMIs (e.g. [6,24,25]).

The final step of the design is the controller *contraction* to the original space. Let K be an $m\times n$ matrix in the state feedback $u=Kx$, chosen in such a way that the closed-loop system in the expanded space includes the closed-loop system in the original space. Then, in the case when $\mathbf{S}$ is a restriction, type (a), of $\tilde{\mathbf{S}}$, K is obtained from the contraction relation $RK=\tilde{K}_DV$, and in the case of restriction, type (b), using $K=Q\tilde{K}_DV$ [22].

3. Stabilization of fixed modes

Our goal is to apply to $\tilde{\mathbf{S}}$ standard control design procedures, such as the LMI-based methods, and by stabilizing $\tilde{\mathbf{S}}$ to *guarantee* stabilization of $\mathbf{S}$ after contraction, having in mind that only then we have a transparent effect of the choice of the overlapping decentralized feedback on stability of the overall contracted system. Feasibility of this approach may be violated by *unstable fixed modes* in $\tilde{\mathbf{S}}$, in spite of the fact that these modes are eliminated after contraction, and the contracted closed-loop system may become stable even when $\tilde{\mathbf{S}}$ is not stabilized (see e.g. [11]).

In order to investigate such a possibility, define a nonsingular matrix W as $W=\left[V \;\vdots\; \hat{V}\right]$, where $\hat{V}$ is defined in (3); consequently, $W^{-1}=\left[\begin{array}{c} U\\ \hdashline \hat{U}\end{array}\right]$. Assuming restriction, type (b), matrix $\tilde{A}+\tilde{B}\tilde{K}_D$, i.e. the closed-loop state matrix in the expanded space, can be transformed in the following way:

$$\begin{aligned} W^{-1}(\tilde{A}+\tilde{B}\tilde{K}_D)W &= \left[\begin{array}{c:c} A+UMV+U\tilde{B}\tilde{K}_DV & AU\hat{V}+UM\hat{V}+U\tilde{B}\tilde{K}_D\hat{V}\\ \hdashline \hat{U}VA+\hat{U}MV+\hat{U}\tilde{B}\tilde{K}_DV & \hat{U}VAU\hat{V}+\hat{U}M\hat{V}+\hat{U}\tilde{B}\tilde{K}_D\hat{V} \end{array}\right]\\ &= \left[\begin{array}{c:c} A+BK & UM\hat{V}+U\tilde{B}\tilde{K}_D\hat{V}\\ \hdashline 0 & \hat{U}M\hat{V} \end{array}\right]. \end{aligned} \tag{8}$$

The obtained result follows from (3), from the fact that $U\hat{V} = 0$ and $\hat{U}V = 0$ by definition, as well as from the relation $U\tilde{B}\tilde{K}_D V = BK$, where $K = Q\tilde{K}_D V$. The block-triangular form of the resulting matrix shows that the expanded closed-loop system contains the closed-loop modes of **S**, together with the modes of $\hat{U}M\hat{V}$, introduced, according to (4), by the expansion of **S** itself. The last modes are fixed: they cannot be influenced by state feedback $\tilde{K}_D$. Obviously, *their instability may prevent control design* in the expanded space; this holds, in particular, for the methods based on LMIs. Therefore, restriction, type (b), guarantees contractibility at the expense of eventually loosing stability caused by instability of the modes introduced by expansion.

A possible idea of how to circumvent this problem could be based on the application of restriction, type (a), having in mind broader possibilities this type of inclusion offers for choosing matrix $\tilde{B}$ from $\tilde{B}R = VB$. Indeed, we can find directly that a convenient choice is here $\tilde{B} = \tilde{B}_D = \text{diag}\{\tilde{B}^1, \ldots, \tilde{B}^N\}$, where $\tilde{B}^i$ is defined in (5). Therefore, we have

$$W^{-1}(\tilde{A} + \tilde{B}_D\tilde{K}_D)W = \left[\begin{array}{c:c} A+BK & UM\hat{V} + U\tilde{B}_D\tilde{K}_D\hat{V} \\ \hdashline 0 & \hat{U}(M+\tilde{B}_D\tilde{K}_D)\hat{V} \end{array}\right] \quad (9)$$

and we conclude that the modes introduced by expansion can now be influenced by the state feedback $\tilde{K}_D$. However, no block-diagonal $\tilde{K}_D$ can be contracted, since in this case the equation $RK = \tilde{K}_D V$ does not have any solution for K [15,22]. Trying to overcome this problem, modifications of the stabilizing $\tilde{K}_D$ have been proposed in [15,18], aimed at satisfying contractibility conditions. However, modified stabilizing feedback gain matrices do not guarantee any more, in general, stability of the closed-loop system in the expanded space, and we remain in a vicious circle.

Another idea to overcome the above contractibility/stability dilemma within the framework of state expansion and stabilization using LMIs has been presented in [25]; it is based on an LMI scheme requiring a careful choice of a number of parameters to be selected a priori.

The idea we advance in this note is, however, simple. We propose that the problem be resolved at the outset of the expansion of **S**. Namely, we propose such an expansion of **S** in which the eigenvalues introduced in the state matrix $\tilde{A}$ of $\tilde{\mathbf{S}}$ are immediately made stable, eliminating the entire problem in advance, and allowing, therefore, application of arbitrary methodologies for control design. The reasoning comes back to (3), which defines the complementary matrix $M = Y(I - VU) = Y\hat{V}\hat{U}$ to be used subsequently in the construction of $\tilde{A}$. Instead of matrix Y, which is initially selected to achieve predefined structural properties of $\tilde{A}$ (see Examples 1 and 2), we propose to introduce $Y + Y_\varepsilon$, where the role of the additional term Y_ε is to stabilize the eigenvalues of $\hat{U}(Y + Y_\varepsilon)\hat{V}$, i.e. all the eigenvalues introduced by expansion. Consequently, if we select the simplest possible structure

$$Y_\varepsilon = -\varepsilon I, \quad (10)$$

we have to choose $\varepsilon \geqslant 0$ in such a way that the matrix $\hat{U}Y\hat{V} - \varepsilon I$ is stabilized. Such an ε always exists: if $\hat{U}Y\hat{V}$ is stable, then $\varepsilon = 0$; if $\hat{U}Y\hat{V}$ is unstable and $\max_i \text{Re}\{\lambda_i(\hat{U}Y\hat{V})\} = \kappa$, then $\hat{U}(Y + Y_\varepsilon)\hat{V}$ is stable for all $\varepsilon > \kappa$ [5]. Obviously, it is also possible to select more complex structures of Y_ε. Notice only that the additional component of the complementary matrix $M_\varepsilon = Y_\varepsilon\hat{V}\hat{U}$ satisfies the inclusion condition $M_\varepsilon V = 0$ for any chosen Y_ε.

Example 3. Assume that $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $B = \text{diag}\{1, 2, 3\}$ in (1). Applying the state transformation as Example 1, one obtains the expansion characterized by

$$\tilde{A} = \left[\begin{array}{cc:cc:cc} 1 & 2 & 0 & 3 & 0 & 0 \\ 4 & 5 & 0 & 0 & 0 & 6 \\ \hdashline 0 & 2 & 1 & 3 & 0 & 0 \\ 0 & 0 & 7 & 9 & 8 & 0 \\ \hdashline 4 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 0 & 8 & 9 \end{array}\right], \qquad \tilde{B} = \frac{1}{2}\left[\begin{array}{cc:cc:cc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ \hdashline 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 3 \\ \hdashline 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 & 0 & 3 \end{array}\right]$$

Choose

$$\hat{V}^{\mathrm{T}} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \hat{U} = \hat{V}^{+} = \tfrac{1}{2}\hat{V}^{\mathrm{T}}$$

according to the definition of V in Example 1 (notice that the structure of $\hat{V}$ follows directly the one of V). According to (4), we obtain that

$$\pi(\hat{U}M\hat{V}) = \pi(\text{diag}\{A_{11}, A_{22}, A_{33}\}) = \pi(\text{diag}\{1, 5, 9\}). \quad (11)$$

Obviously, the modes of $\tilde{A}$ introduced by the expansion coincide with the modes of the diagonal blocks where subsystems $\tilde{\mathbf{S}}^i$ overlap, i.e. they are placed at 1, 5 and 9. According to the proposed strategy, we modify the obtained $\tilde{A}$ by introducing the following additional complementary matrix:

$$M_\varepsilon = -\varepsilon\hat{V}\hat{U} = -\frac{1}{2}\varepsilon \left[\begin{array}{cc:cc:cc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \hdashline 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \hdashline 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

where $\varepsilon > 9$; obviously, $M_\varepsilon V = 0$.

Example 4. Let $A = \text{diag}\{-1, 1, -1\}$ and $B = [1 \quad b \quad 1]^{\mathrm{T}}$. Applying the transformation from Example 2, one obtains that the resulting expanded state matrix $\tilde{A} = \text{diag}\{-1, 1, 1, -1\}$ has a fixed mode at 1. It is obvious that the complementary matrix aimed at stabilizing this mode is given by

$$M_\varepsilon = \frac{1}{2}\varepsilon \left[\begin{array}{cc:cc} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ \hdashline 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

where $\varepsilon > 1$, having in mind that $\hat{V} = [0 \quad -1 \quad 1 \quad 0]^{\mathrm{T}}$ and $\hat{U} = \frac{1}{2}\hat{V}^{\mathrm{T}}$, so that the modes of the expansion become $\{-1, 1, 1-\varepsilon, -1\}$. The fixed mode introduced by expansion is $1-\varepsilon$; the remaining mode at 1 is not fixed, provided the original system is stabilizable: it can be influenced by the state feedback if $b \neq 0$.

Further analysis of the matrix $M_\varepsilon = Y_\varepsilon \hat{V}\hat{U} = -\varepsilon\hat{V}\hat{U}$ involved in the proposed stabilization procedure shows that, in general, the nonzero blocks of $\hat{V}\hat{U} = I - VU$ appear always exactly at the places of the overlapping diagonal elements A_{kk} of A in the matrix VAU. Namely, if the block A_{kk} appears $\nu = 1$ times in VAU, then it is not an overlapping block, and $\hat{V}\hat{U}$ contains zero at the corresponding place. However, if A_{kk} appears $\nu > 1$ times at the main block-diagonal of VAU, then it appears ν^2 times in VAU, and ν^2 nonzero blocks in $\hat{V}\hat{U}$ corresponding to A_{kk} compose the following structure:

$$\frac{1}{\nu}\begin{bmatrix} (\nu-1)I & -I & \dots & -I \\ -I & (\nu-1)I & \dots & -I \\ \dots & & & \\ -I & \dots & & (\nu-1)I \end{bmatrix}.$$

Consequently, the proposed method based on (10) can be interpreted as a method for eliminating unstable fixed modes of $\tilde{A}$ by modifying the overlapping diagonal blocks of A. We can simply see that in Example 1 we have $\nu = 2$ for all three overlapping blocks A_{kk}, $(k = 1, 2, 3)$, while we have $\nu = 2$ only for the overlapping block A_{22} in Example 2.

Obviously, the matrix M_ε introduces new elements in the matrix $\tilde{A}$ that are at the first glance far from being negligible, especially for the modes far in the right half plane. The influence of M_ε on the stabilizing feedback design is found not to be deteriorating as long as the introduced fixed modes are not too close to the imaginary axis (which can be easily avoided). The following example gives an insight into the influence of the parameter ε; it will be shown that it can have even additional positive effects on the system performance.

Example 5. Applicability of the proposed approach to the stabilization of fixed modes will be further illustrated on the numerical example already used in [25]. The system is described by

$$\mathbf{S}: \ \dot{x} = Ax + Bu = \begin{bmatrix} 1 & 4 & 0 \\ 1 & 2 & 2 \\ 0 & -2 & 3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ b & 0 \\ 0 & 1 \end{bmatrix} u,$$
$$x(t_0) = x_0. \qquad (12)$$

Assuming $\mu = 2$, $N_1 = N_2 = 2$, $l_1^1 = 1$, $l_2^1 = 2$, $l_1^2 = 2$ and $l_2^2 = 3$ one can produce an expansion $\tilde{\mathbf{S}}$ of restriction type characterized by

$$\tilde{A} = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 1 & 2-\varepsilon' & \varepsilon' & 2 \\ 1 & \varepsilon' & 2-\varepsilon' & 2 \\ 0 & 0 & -2 & 3 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 1 & 0 \\ b & 0 \\ b & 0 \\ 0 & 1 \end{bmatrix}$$

using transformations $v = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$ and $R = I$, and introducing the proposed stabilization of fixed modes according to (10), with $\hat{V} = [0 \quad -1 \quad 1 \quad 0]^{\mathrm{T}}$, $\hat{U} = \frac{1}{2}\hat{V}^{\mathrm{T}}$ and $\varepsilon' = \frac{1}{2}\varepsilon$. Obviously, the resulting fixed modes are the eigenvalues of

$$\hat{U}(M + M_\varepsilon)\hat{V} = \tfrac{1}{2}[0 \quad -1 \quad 1 \quad 0]\left(\begin{bmatrix} 0 & 2 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\varepsilon' & \varepsilon' & 0 \\ 0 & \varepsilon' & -\varepsilon' & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = 2 - \varepsilon,$$

having in mind that

$$M = \begin{bmatrix} 0 & A_{12} & -A_{12} & 0 \\ 0 & A_{22} & -A_{22} & 0 \\ 0 & -A_{22} & A_{22} & 0 \\ 0 & -A_{32} & A_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

according to Example 2 and [18].

Taking $b = 1$ and $\varepsilon = 2.1$, the design procedure based on LMI stabilization of $\tilde{\mathbf{S}}$ by block diagonal state feedback matrix with 1×2 diagonal blocks and contraction back to the original space results into $K = \begin{bmatrix} -160 & -5.15 & 0 \\ 0 & -1.91 & -7.4 \end{bmatrix}$, with the closed-loop modes $\{-0.45, -3.86 \pm \mathrm{j}2.69\}$. Increasing ε one gets the following closed-loop modes: $\{-0.42, -4.35 \pm \mathrm{j}5.62\}$ for $\varepsilon = 5$, $\{-0.31, -11.65 \pm \mathrm{j}11.25\}$ for $\varepsilon = 10$ and $\{-0.19, -0.4292 \pm \mathrm{j}45.55\}$ for $\varepsilon = 50$.

Taking $b = 0$ one obtains, according to [25], the Type II overlapping, and the stabilization problem in the original model space becomes infeasible. Applying the above proposed expansion with preliminary stabilization of fixed modes, one obtains directly, in the same way as above, the following closed-loop modes: $\{-3.36, -3.09 \pm \mathrm{j}2.48\}$ for $\varepsilon = 2.1$, $\{-4.14, -2.03 \pm \mathrm{j}6.08\}$ for $\varepsilon = 5$, $\{-6.97, -5.19 \pm \mathrm{j}10.77\}$ for $\varepsilon = 10$ and $\{-16.69, -31.23 \pm \mathrm{j}48.73\}$ for $\varepsilon = 50$. Obviously, by increasing ε one shifts the closed-loop modes to the left in the s-plane, getting a faster response (at the price of possible violation of the linearity assumption).

Notice that the proposed procedure is much simpler than the one proposed in [25], which is based, in the given example, on three additional parameters to be found by trial and error.

4. Conclusion

In this note an efficient general method for stabilization of fixed modes in expansions of LTI systems is proposed. Starting from the expansion/contraction paradigm and the inclusion principle, it is shown that overlapping decentralized control design based on expansions satisfying restriction conditions can suffer from the problem of instability of fixed modes. A precise elaboration leads to the formulation of a general, simple and efficient method for overcoming this problem, enabling, in such a way, broader possibilities for control design methods based on generalized overlapping decompositions. Characteristic numerical examples illustrate properties of the proposed method.

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