Stabilization of Nonlinear Systems With Moving Equilibria

A. I. Zecevic and D. D. Siljak

Abstract—**This note provides a new method for the stabilization of nonlinear systems with parametric uncertainty. Unlike traditional techniques, our approach does not assume that the equilibrium remains fixed for all parameter values. The proposed method combines different optimization techniques to produce a robust control that accounts for uncertain parametric variations, and the corresponding equilibrium shifts. Comparisons with analytical gain scheduling are provided.**

Index Terms—**Linear matrix inequalities, moving equilibria, nonlinear optimization, parametric stability, robustness.**

I. Introduction

In the analysis of nonlinear dynamic systems, it is common practice to separately treat the existence of equilibria and their stability. The traditional approach has been to compute the equilibrium of interest, and then introduce a change of variables that translates the equilibrium to the origin. This methodology has been widely applied to systems that contain parametric uncertainties, and virtually all control schemes developed along these lines implicitly assume that the equilibrium remains fixed for the entire range of parameter values [1]–[5].

It is important to note, however, that there are many practical applications where the fixed equilibrium assumption is not realistic. In fact, it is often the case that variations in the system parameters result in a moving equilibrium, whose stability properties can vary substantially. In some situations, the equilibrium could even disappear altogether, as in the case of heavily stressed electric power systems [6]–[8]. Much of the recent work involving moving equilibria has focused on analytical gain scheduling [9]–[12]. This approach assumes the existence of an exogenous scheduling variable, whose instantaneous value determines the appropriate control law (which may be nonlinear in general). Analytical gain scheduling will be discussed in some detail in Section IV, where it is compared with the method proposed in this note.

For our purposes, it is suitable to use the concept of parametric stability, which simultaneously captures the *existence* and the *stability* of a moving equilibrium [13]–[17]. This concept has been formulated in [14], where a general nonlinear dynamic system

$$\dot{x} = f(x, p) \tag{1}$$

was considered, with the assumption that a stable equilibrium state $x^e(p^*) \in R^n$ corresponds to the nominal parameter value $p = p^* \in R^l$. System (1) is said to be *parametrically stable* at p^* if there is a neighborhood $\Omega(p^*) \subset R^l$ such that

i) an equilibrium $x^e(p) \in R^n$ exists for any $p \in \Omega(p^*)$;
ii) equilibrium $x^e(p)$ is stable for any $p \in \Omega(p^*)$.

With this definition in mind, the main objective of this note will be to develop a strategy for the parametric stabilization of nonlinear systems. Our approach combines two different optimization techniques to produce a robust control that allows for unpredictable equilibrium shifts due to parametric variations. The resulting controller is linear, and the corresponding gain matrix is obtained using linear matrix inequalities (LMIs) [18]–[23]. The reference input values, on the other hand, are computed by a nonlinear constrained optimization procedure that takes into account the sensitivity of the equilibrium to parameter changes.

The note is organized as follows. In Section II, we provide a brief overview of the control design using linear matrix inequalities, and extend these concepts to systems with parametrically dependent equilibria. Section III is devoted to the problem of selecting an appropriate reference input, and the effects that this selection may have on the size of the stability region in the parameter space. The proposed control strategy is then compared with analytical gain scheduling in Section IV.

II. Parametric Stabilization Using Linear Matrix Inequalities

Let us consider a general nonlinear system described by the differential equations

$$\dot{x} = Ax + h(x) + Bu \tag{2}$$

where $x \in R^n$ is the state of the system, $u \in R^m$ is the input vector, A and B are constant $n \times n$ and $n \times m$ matrices, and h: $R^n \rightarrow R^n$ is a piecewise-continuous nonlinear function in x, satisfying $h(0) = 0$. The term $h(x)$ is assumed to be uncertain, but bounded by a quadratic inequality

$$h^T h \leq \alpha^2 x^T H^T H x \tag{3}$$

where $\alpha > 0$ is a scalar parameter and H is a constant matrix. In the following, it will be convenient to rewrite this inequality as:

$$\begin{bmatrix} x \\ h \end{bmatrix}^T \begin{bmatrix} -\alpha^2 H^T H & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ h \end{bmatrix} \leq 0. \tag{4}$$

If we assume a linear feedback control law $u = Kx$, the closed-loop system takes the form

$$\dot{x} = \hat{A}x + h(x) \tag{5}$$

where $\hat{A} = A + BK$. The global asymptotic stability of (5) can then be established using a Lyapunov function

$$V(x) = x^T P x \tag{6}$$

where P is a symmetric positive-definite matrix (denoted $P > 0$). As is well known, a sufficient condition for stability is for the derivative of $V(x)$ to be negative along the solutions of (5). Formally, this condition can be expressed as a pair of inequalities

$$P > 0, \quad \begin{bmatrix} x \\ h \end{bmatrix}^T \begin{bmatrix} \hat{A}^T P + P\hat{A} & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x \\ h \end{bmatrix} < 0. \tag{7}$$

Defining $Y = \tau P^{-1}$ (where τ is a positive scalar), $L = KY$, and $\gamma = 1/\alpha^2$, the control design can now be formulated as an LMI problem in Y, L and γ [22]:

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The authors are with the Department of Electrical Engineering, Santa Clara University, Santa Clara, CA 95053 USA.

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Problem 1: Minimize γ, subject to $Y > 0$ and

$$\begin{bmatrix} AY + YA^T + BL + L^T B^T & I & YH^T \\ I & -I & 0 \\ HY & 0 & -\gamma I \end{bmatrix} < 0. \quad (8)$$

If the optimization problem (8) is feasible, the resulting gain matrix stabilizes system (5) for *all* nonlinearities satisfying (3). We should note, however, that the conditions in (8) place no restrictions on the size of the gain. To limit the gain and, at the same time, guarantee a desired value $\overline{\alpha}$, we need to apply the following modification of the optimization problem [22].

Problem 2: Minimize $c_1\gamma + c_2\kappa_Y + c_3\kappa_L$ subject to $Y > 0$,

$$\begin{bmatrix} AY + YA^T + BL + L^T B^T & I & YH^T \\ I & -I & 0 \\ HY & 0 & -\gamma I \end{bmatrix} < 0 \quad (9)$$

and

$$\gamma - 1/\overline{\alpha}^2 < 0 \quad \begin{bmatrix} -\kappa_L I & L^T \\ L & -I \end{bmatrix} < 0 \quad \begin{bmatrix} Y & I \\ I & \kappa_Y I \end{bmatrix} > 0 \quad (10)$$

where c_1, c_2 and c_3 are appropriate positive weighting factors.

The variables $\kappa_Y > 0$ and $\kappa_L > 0$ represent constraints on the size of the gain K, since $\|K\| \leq \kappa_Y\sqrt{\kappa_L}$ (where $\|\cdot\|$ is the Euclidean norm). This relationship is easily derived by observing that (10) implies

$$L^T L < \kappa_L I, \quad Y^{-1} < \kappa_Y I \quad (11)$$

and that $K = LY^{-1}$ by definition. With that in mind, it can be said that the LMI procedure indirectly minimizes $\|K\|$, subject to constraints (9) and (10). The precise nature of this minimization depends, of course, on the weighting factors (in our simulations we found that $c_1 = 0.01$, $c_2 = 10$ and $c_3 = 0.01$ is an appropriate choice for a wide range of problems). We should also point out that if a specific bound on $\|K\|$ is desired, it is typically necessary to iteratively adjust $\overline{\alpha}$ until such a gain matrix is obtained. This is a simple procedure that requires only a few steps, given that $\overline{\alpha}$ is a scalar quantity.

In the following, we will be interested in extending these results to systems of the form:

$$\dot{x} = Ax + h(x, p) + Bu \quad (12)$$

where $p \in R^l$ is an uncertain parameter vector, and the control has the general form

$$u = r + K(x - x^r) \quad (13)$$

in which r and x^r denote user-defined reference variables. Unlike the conventional approach to stability under parametric uncertainty, we will assume that the equilibrium of the closed-loop system, $x^e(p)$, is *not* confined to the origin. To develop an appropriate mathematical framework for this type of problem, let us introduce a new state vector y, defined as the deviation of state x from the equilibrium $x^e(p)$

$$y \equiv x - x^e(p). \quad (14)$$

Since the equilibrium must satisfy

$$Ax^e(p) + h(x^e(p), p) + B\left[r + K(x^e(p) - x^r)\right] = 0 \quad (15)$$

it is straightforward to show that the proposed change of variables eliminates r and x^r from the model, producing a closed-loop system

$$\dot{y} = (A + BK)y + g(x^e(p), p, y) \quad (16)$$

with

$$g(x^e(p), p, y) \equiv h(x^e(p) + y, p) - h(x^e(p), p). \quad (17)$$

At this point, we need to introduce three key assumptions.

Assumption 1: The variation of parameter p is limited to a ball Ω centered around some nominal value p^*

$$\Omega = \left\{p \in R^l \,|\, \|p - p^*\| \leq \rho\right\}. \quad (18)$$

Assumption 2: The closed-loop equilibrium $x^e(p)$ is a continuous functions of p for all $p \in \Omega$. Its nominal value $x^e(p^*)$ will be denoted in the following by x^*.

Assumption 3: The function $g(x^e(p), p, y)$ can be bounded in such a way that inequality

$$g^T g \leq y^T H^T(x^e(p), p) H(x^e(p), p) y \quad (19)$$

holds for some matrix $H(x^e(p), p)$ whose elements are continuous function of $x^e(p)$ and p.

Using Assumptions 1–3, it can be shown that there exists a constant $\mu > 0$ such that

$$y^T H^T(x^e(p), p) H(x^e(p), p) y \leq \mu\, y^T H(x^*, p^*)^T H(x^*, p^*) y \quad (20)$$

for any $p \in \Omega$, provided that matrix $H(x^*, p^*)$ is nonsingular. Whenever this is the case, matrix $H(x^*, p^*)$ can be used in place of H in the LMI optimization (9). We should note in this context that if $H(x^e(p), p)$ is a *diagonal* matrix, it is possible to obtain an explicit estimate of region Ω.

It is also important to recognize a problem that arises in the determination of the nominal equilibrium point x^*. Namely, from (15) it follows that x^* depends on r, x^r and K. On the other hand, the construction of matrix $H(x^*, p^*)$ (and, therefore, the computation of K as well) requires prior knowledge of x^*. To avoid this circularity, we propose to determine the reference input x^r in the following way:

Step 1) Fix r and p^*, and solve

$$Ax + h(x, p^*) + Br = 0 \quad (21)$$

for x^*. This can be done using Newton's method, under relatively mild assumptions on h (e.g., [24]).

Step 2) Set $x^r = x^*$. In that case, x^* will be the equilibrium of the closed-loop system

$$\dot{x} = Ax + h(x, p^*) + B\left[r + K(x - x^*)\right] \quad (22)$$

for *any* choice of K.

The following simple example serves to illustrate the main ideas behind the proposed approach.

Example 1: Let us consider the system

$$\begin{aligned} \dot{x}_1 &= 3x_1 + x_2 + x_1 \sin x_2 + p - 2 + u \\ \dot{x}_2 &= -x_1 + 0.5x_2 + \sin^2 x_1 + p - 3 + u \end{aligned} \quad (23)$$

in which the nonlinear term

$$h(x, p) = \begin{bmatrix} x_1 \sin x_2 + p - 2 \\ \sin^2 x_1 + p - 3 \end{bmatrix} \quad (24)$$

explicitly depends on parameter p. For simplicity, we will assume that $r = 0$ and $p^* = 0$, which yields $x^* = [-0.7094\ 3.7328]^T$. It is easily verified that this equilibrium is unstable in the absence of control.

Using the trigonometric identity

$$\sin\beta - \sin\gamma = 2\cos\left(\frac{\beta+\gamma}{2}\right)\sin\left(\frac{\beta-\gamma}{2}\right) \tag{25}$$

it can be shown that the components of $g(x^e(p), p, y)$ satisfy

$$\begin{aligned} g_1(x^e(p), p, y) &= 2x_1^e(p)\cos(x_2^e(p) + y_2/2)\sin(y_2/2) \\ &\quad + y_1 \sin(x_2^e(p) + y_2) \\ g_2(x^e(p), p, y) &= \sin(2x_1^e(p) + y_1)\sin(y_1). \end{aligned} \tag{26}$$

This function can now be bounded as proposed in (19), using

$$H(x^e(p), p) = \begin{bmatrix} [2(1+|x_1^e(p)|)]^{1/2} & 0 \\ 0 & \left[|x_1^e(p)|^2 + 0.5\,|x_1^e(p)|\right]^{1/2} \end{bmatrix}. \tag{27}$$

Observing that $x_1^* = -0.7094$, we obtain $H_{11}(x^*, p^*) = 1.849$ and $H_{22}(x^*, p^*) = 0.926$, respectively. Setting $\|K\| \le 25$ as a desired bound on the gain norm, the LMI optimization now produces $\alpha = 1.25$ and $K = [-22.3 \;\; 9.2]$.

Using the fact that the closed-loop system is guaranteed to be stable for *any* nonlinearity $\tilde{g}(y)$ that satisfies

$$\begin{aligned} \tilde{g}^T\tilde{g} &\le \alpha^2 y^T H(x^*, p^*)^T H(x^*, p^*) y \\ &= \alpha^2 y^T \begin{bmatrix} 3.419 & 0 \\ 0 & 0.858 \end{bmatrix} y \end{aligned} \tag{28}$$

and recognizing that $H(x^e(p), p)$ is a diagonal matrix, it follows that the maximal permissible value of $x_1^e(p)$ must satisfy the following pair of inequalities:

$$\begin{aligned} H_{11}^2(x^e(p), p) &= 2(1+|x_1^e(p)|) \le 3.419\alpha^2 \\ H_{22}^2(x^e(p), p) &= |x_1^e(p)|^2 + 0.5\,|x_1^e(p)| \le 0.858\alpha^2. \end{aligned} \tag{29}$$

It is easily established that, for $\alpha = 1.25$, (29) holds when $|x_1^e(p)| \le 0.9345$, and that $|x_1^e(p)| = 0.9345$ for $p = -17.4$. We can therefore conclude that the closed-loop system remains stable as long as $|p| \le 17.4$.

III. Selection of the Reference Input

In Example 1, the reference input r was chosen arbitrarily. It should be noted, however, that this quantity provides an additional degree of freedom in the design, and some effort should be made to choose it in a systematic manner. The following observations will prove to be helpful in identifying the objectives that an optimal choice of r ought to achieve.

1) The design proposed in Section II assumes that the minimization problem (9) and (10) is feasible. To ensure this, it is desirable to choose x^* so that $\|H(x^*, p^*)\|$ is as small as possible.
2) Our simulations have shown that a reduction in $\|H(x^*, p^*)\|$ generally results in a larger value for α, given a fixed bound on the gain norm.
3) In Example 1 it was established that if $H(x^e(p), p)$ is a diagonal matrix with entries $H_{ii}(x^e(p), p)$, the stability region Ω in the parameter space can be estimated using

$$H_{ii}(x^e(p), p) \le \alpha^2 H_{ii}(x^*, p^*) \qquad i = 1, 2, \ldots, n. \tag{30}$$

From that standpoint, a larger α will clearly allow for a larger range of permissible values for $x^e(p)$. It should be noted, however, that an improved range for $x^e(p)$ does *not* automatically guarantee a corresponding increase in the size of Ω. For this to be the case, it is also necessary for the sensitivity matrix $\partial x^e(p)/\partial p$ to be appropriately bounded.

In view of these remarks, we now propose a strategy for selecting a reference input r that minimizes $\|H(x^*, p^*)\|$ while imposing certain constraints on the sensitivity of the equilibrium. We begin by observing that the equilibrium $x^e(p)$ represents the solution of

$$Ax + h(x, p) + Br + BK(x - x^*) = 0. \tag{31}$$

It is clear from (31) that $x^e(p)$ implicitly depends on the choice of reference input r. Differentiating (31) with respect to p, we now obtain the following expression for the sensitivity of the equilibrium:

$$\xi(p, K) \equiv \frac{\partial x^e(p)}{\partial p} = -\left[(A + BK) + \frac{\partial h}{\partial x}\right]^{-1}\frac{\partial h}{\partial p}. \tag{32}$$

Ideally, one would like to determine a value for r that minimizes $\|\xi(p, K)\|$ over the widest possible range of parameter values. This, however, is a difficult problem, which we will not attempt to solve in this note. Instead, we will take a more heuristic approach and focus on the special case when p takes its nominal value, p^*. The sensitivity vector that corresponds to p^* can be expressed as

$$\xi^*(K) = \xi(p^*, K) = -J(K)^{-1}\left.\frac{\partial h}{\partial p}\right|_{p^*} \tag{33}$$

where $J(K)$ is the Jacobian

$$J(K) = A + BK + \left.\frac{\partial h}{\partial x}\right|_{(x^*, p^*)}. \tag{34}$$

Before proceeding, it is important to recognize that input r can not be used to minimize $\|\xi^*(K)\|$ directly, since the gain matrix K is *not* known at the time when r is computed; this is evident from (21) and (22), and the related discussion in Section II. In order to address this problem, let us observe that the sensitivity vector without control satisfies

$$\xi^*(0) = -J(0)^{-1}\left.\frac{\partial h}{\partial p}\right|_{p^*}. \tag{35}$$

Utilizing the Sherman–Morrison formula (e.g., [25]), we now obtain

$$\xi^*(K) = \left[I - Q(I_m + KQ)^{-1}K\right]\xi^*(0) \tag{36}$$

where $Q \equiv J(0)^{-1}B$ and I_m is an $m \times m$ identity matrix (where m represents the number of inputs in the system). In most practical problems $m \ll n$, so the matrix inversion in (36) can be performed easily.

It is clear from (34) and (35) that the sensitivity matrix $\xi^*(0)$ depends only on the nominal equilibrium x^*, which can be modified by an appropriate selection of r. The same holds true for matrix $H(x^*, p^*)$. With that in mind, a logical design strategy would be to determine an input r that simultaneously minimizes $\|H(x^*, p^*)\|$ and $\|\xi^*(0)\|$. In this way, we would indirectly bound $\|\xi^*(K)\|$ as well, by virtue of (36). The computation of such an r can now be formulated as the following constrained optimization problem.

Problem 3: Minimize $\Phi(x^*) \equiv c_1\|\xi^*(0)\| + c_2\|H(x^*, p^*)\|$ subject to

$$\left[A + \frac{\partial h}{\partial x}\right]\xi^*(0) + \frac{\partial h}{\partial p} = 0 \tag{37}$$

$$Ax + h(x, p^*) + Br = 0 \tag{38}$$

$$H_{ii}(x^*, p^*) > 0.1, \qquad i = 1, 2, \ldots, n. \tag{39}$$

The weighting factors c_1 and c_2 in the cost function $\Phi(x^*)$ are to some extent problem dependent, and reflect the relative importance

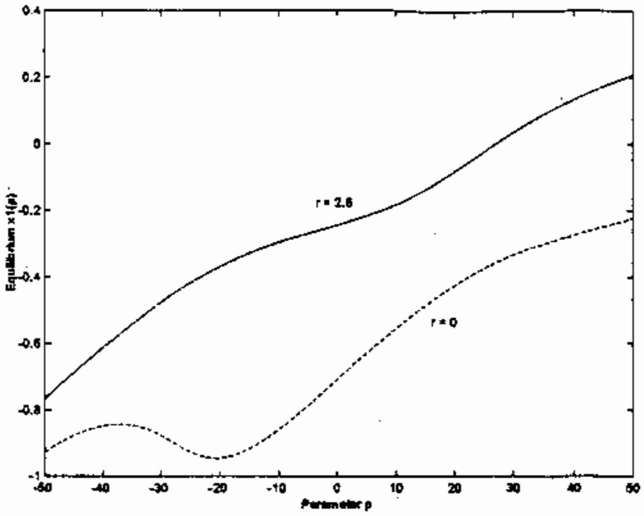

Fig. 1. Equilibrium $x_1^e(p)$ for $r = 0$ and $r = 2.6071$.

of minimizing $\|H(x^*, p^*)\|$ and satisfying the sensitivity constraints. The additional constraints in (39) are included to prevent matrix $H(x^*, p^*)$ from becoming singular, which could render the LMI optimization infeasible.

The optimization described by (37)–(39) should be the very first step in the design process, as noted in Section II. The following example demonstrates how such an approach can enhance the parametric stability of the system.

Example 2: Let us once again consider the nonlinear system (23), for which (37) and (38) take the form

$$\begin{aligned}(3 + \sin x_2)\,\xi_1 + (1 + x_1 \cos x_2)\,\xi_2 + 1 &= 0\\ (\sin 2x_1 - 1)\,\xi_1 + 0.5\xi_2 + 1 &= 0\end{aligned} \tag{40}$$

and

$$\begin{aligned}3x_1 + x_2 + x_1 \sin x_2 - 2 + r &= 0\\ -x_1 + 0.5x_2 + \sin^2 x_1 - 3 + r &= 0\end{aligned} \tag{41}$$

respectively. The optimization procedure with weighting factors $c_1 = 1$ and $c_2 = 5$ produces $r = 2.6071$ and a nominal equilibrium $x^* = [-0.2463 \quad 0.1745]^T$. Since the value of x^* is now different from the one in Example 1, a new LMI procedure needs to be performed, this time with

$$H(x^*, p^*) = \begin{bmatrix} 1.5788 & 0 \\ 0 & 0.4287 \end{bmatrix}. \tag{42}$$

Using $\|K\| \leq 25$ as a constraint, we obtain $\alpha = 1.95$ and $K = [-23.3 \quad 9.1]$, respectively. It should be noted that the value of α is considerably larger than the one obtained in Example 1 for the same bound on $\|K\|$.

In this case, inequalities

$$\begin{aligned}2\,(1 + |x_1^e(p)|) &\leq \alpha^2 H_{11}^2(x^*, p^*) = 9.478\\ |x_1^e(p)|^2 + 0.5\,|x_1^e(p)| &\leq \alpha^2 H_{22}^2(x^*, p^*) = 0.6988\end{aligned} \tag{43}$$

imply that the system is stable whenever $|x_1^e(p)| \leq 0.6225$. To see what this means from the standpoint of parametric stability, in Fig. 1 we show the evolution of $x_1^e(p)$ for $r = 0$ and $r = 2.6071$, with $\|K\| \leq 25$ in both cases. It is readily observed that the critical parameter value is $p = -40.5$ for the optimized case, compared to $p = -17.4$ for the case when $r = 0$ (as shown in Example 1). This indicates a significant improvement in the size of the stability region due to the choice of the reference input.

IV. Comparisons With Analytical Gain Scheduling

In this section, we provide a comparison between the proposed optimization-based control and analytical gain scheduling, which is an alternative method for stabilizing systems with moving equilibria. The mathematical framework for gain scheduling assumes a system description of the form [9]

$$\begin{aligned}\dot{x} &= f(x(t),\, u(t),\, p(t))\\ y &= \varphi(x(t),\, u(t),\, p(t))\end{aligned} \tag{44}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the input, $y(t) \in R^q$ is the output, and $p(t) \in R^l$ is an exogenous scheduling variable, which is available for control purposes. In general, the control law

$$u(t) = k(x(t),\, p(t)) \tag{45}$$

is assumed to be nonlinear, and dependent on both $x(t)$ and $p(t)$.

For any steady-state value $p \in \Omega \subset R^l$ of the scheduling variable, we can compute the corresponding equilibrium $x^e(p)$ and a constant control $u(p)$ such that

$$\begin{aligned}f(x^e(p),\, u(p),\, p) &= 0\\ y(p) &= \varphi(x^e(p),\, u(p),\, p).\end{aligned} \tag{46}$$

The function $y(p)$ in (46) is given, and is referred to as the *output trim condition*. In the ideal case, when functions $x^e(p)$ and $u(p)$ can be obtained analytically, (44) can be linearized as

$$\Delta\dot{x} = A(p)\Delta x + B(p)\Delta u + E(p)\Delta p \tag{47}$$

where

$$A(p) \equiv \left.\frac{\partial f}{\partial x}\right|_p \quad B(p) \equiv \left.\frac{\partial f}{\partial u}\right|_p \quad E(p) \equiv \left.\frac{\partial f}{\partial p}\right|_p \tag{48}$$

and

$$\begin{aligned}&\Delta x(t) \equiv x(t) - x^e(p) \quad \Delta u(t) \equiv u(t) - u(p)\\ &\Delta p(t) \equiv p(t) - p\end{aligned} \tag{49}$$

represent deviations from the equilibrium values.

The linearized control will have the form

$$\Delta u(t) = K_1(p)\Delta x(t) + K_2(p)\Delta p(t) \tag{50}$$

with

$$K_1(p) = \left.\frac{\partial k}{\partial x}\right|_p \quad K_2(p) = \left.\frac{\partial k}{\partial p}\right|_p. \tag{51}$$

To ensure local stability, the closed-loop matrix $A(p) + B(p)K_1(p)$ must have eigenvalues in the left half plane for all $p \in \Omega$. In the following, we will assume that an appropriate gain matrix $K_1(p)$ can be determined as an explicit function of p. The nonlinear control law (45) can now be constructed so that the following two conditions are satisfied:

$$k(x^e(p),\, p) = u(p) \tag{52}$$

$$\left.\frac{\partial k}{\partial x}\right|_p = K_1(p). \tag{53}$$

The first condition secures that $u(t)$ has the appropriate steady state value conforming to the output trim condition, while (53) guarantees that the eigenvalues of the linearized system are the desired ones.

As pointed out in [9], there are many functions $k(x(t), p(t))$ that are capable of satisfying conditions (52) and (53). A simple choice would be

$$k(x(t), p(t)) = K_1(p(t))x(t) + [u(p(t)) - K_1(p(t))x^e(p(t))] \quad (54)$$

which represents a combination of linear state feedback and a bias term [both of which explicitly depend on $p(t)$]. It should be noted that when the control is additive and $p(t)$ varies much more slowly than any of the states, the system dynamics can be approximated as

$$\dot{x} = f(x(t), p) + Bu(t) \quad (55)$$

and the control takes the simplified form

$$u(t) \approx K_1(p)[x(t) - x^e(p)] + u(p). \quad (56)$$

Under such circumstances, the gain scheduling method and the optimization-based approach described in the previous sections have basically the same objective: to stabilize (55) for all constant values $p \in \Omega$. From that standpoint, the two control strategies can be compared as follows.

1) The nonlinear model (44) treated by the gain scheduling method is very general, while the LMI approach requires an additive nonlinearity that satisfies (19).
2) Analytic gain scheduling assumes that $p(t)$ is readily available at any point in time, and can be used in computing the control. In our optimization approach only the nominal value p^* is required for control design, and $p(t)$ need not be known explicitly.
3) Idealized analytic gain scheduling requires the existence of explicit expressions for $x^e(p)$, $u(p)$ and $K_1(p)$, which is seldom the case in practice. More realistically, these quantities can be computed for a discrete set of points $p_1, p_2, \ldots, p_n$, and functions $\hat{x}^e(p)$, $\hat{K}_1(p)$ and $\hat{u}(p)$ can then be constructed by interpolation. In this context, special care must be taken to use an interpolation method that guarantees stability for all intermediate points. Two such methods have recently been proposed by Stilwell and Rugh [12]. In contrast to this approach, the LMI-based control utilizes a *constant* gain matrix and *fixed* reference vectors x^* and r. All of these quantities can be computed offline, and require only the nominal parameter value p^*.
4) Since the gain matrix $K_1(p)$ is obtained from a linearized system, the resulting control is necessarily local. The LMI approach does not have that restriction, since it deals directly with the nonlinear model. Furthermore, when matrix $H(x^e(p), p)$ is diagonal, it is possible to explicitly determine a region of stability in the parameter space.
5) The functions $x^e(p)$ and $u(p)$ in (56) are determined by combining the equilibrium equations and an output trim condition. In the LMI approach, there are no preassigned requirements for the output, and the corresponding quantities x^* and r are obtained from an optimization procedure in which the equilibrium equations represent a constraint.

V. Conclusion

In this note, a method was proposed for the stabilization of nonlinear systems with moving equilibria. The control design is based on linear matrix inequalities, and is applicable to systems consisting of a linear part and an additive nonlinearity which can be bounded by a parametrically dependent quadratic form. For a class of such problems, it is possible to explicitly estimate the stability region in the parameter space. It was demonstrated that the size of this region can be increased substantially by an optimal choice of the reference input.

The proposed technique was compared to analytical gain scheduling, which represents an alternative method for controlling systems with moving equilibria. Although the two approaches are based on rather different assumptions, they can be compared in a meaningful way in cases when the scheduling variable changes slowly.

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